Chapter 14

Exotic Spheres

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In Chapter 12, the role played by exotic spheres in the detection and cancellation of global anomalies was extensively analyzed. The purpose of this present chapter is to give a resumé (in the signature case) of the mathematical background involving characteristic classes that implies the existence of exotic spheres. To this end, we first review some basic facts about Chern classes, Pontrjagin classes, and the Hirzebruch index theorem. These facts are then marshalled to prove the existence of exotic spheres; in particular, the Milnor seven-sphere, Σ, and its relatives (see [1] for more information).

First, recall the infinite complex projective space $\mathbb{CP}^\infty$ and its interpretations for line bundles and cohomology: Let $[X, \mathbb{CP}^\infty]$ denote the homotopy classes of mappings of a space X to $\mathbb{CP}^\infty$. Then this homotopy set is isomorphic with the second cohomology group of X:

$$H^2(X) \cong \left[X, \mathbb{CP}_\infty\right].$$

This follows from the fact that $\mathbb{CP}_\infty$ is a $K(\mathbb{Z}, 2)$, a space whose homotopy groups all vanish except for a $\mathbb{Z}$ in dimension two.

It follows from the construction of $\mathbb{CP}^\infty$ that $\left[X, \mathbb{CP}^2\right] \cong \mathcal{L}(X)$, the isomorphism classes of complex line bundles over X. In this case, we have the canonical line bundle Λ over $\mathbb{CP}^\infty$, and a map $f : X \to \mathbb{CP}^\infty$ induces a line bundle $f^* \Lambda$ over X:

$$\begin{array}{ccc} f^*\Lambda & \longrightarrow & \Lambda \\ \downarrow \pi & & \downarrow \pi^* \\ X & \xrightarrow{f} & \mathbb{CP}^\infty. \end{array}$$

If $i \in H^2\left(\mathbb{CP}^\infty\right)$ denotes the generator of the cohomology ring of $\mathbb{CP}^\infty$, then the first Chern class of $f^*\Lambda$, $c_1(f^*\Lambda)$, is found by taking the pull-back of i via f:

$$c_1(f^*\Lambda) = f^*(i) \in H^2(X).$$

It is also not hard to see that $\mathcal{L}(X) = H^2(X)$ as groups, with tensor product of line bundles corresponding to addition in H^2. The first Chern class, c_1, can be interpreted as the self-intersection number of the 0-section of the corresponding bundle.

More generally, let $E \xrightarrow{p} B$ be a complex vector bundle. Then, there exist Chern classes $c_i(E) \in H^{2i}(B;\mathbb{Z})$ satisfying the following properties:

14.0.1 Properties of Chern Classes

(0) $c_i(E) = 0$ for $i > n$ = complex fiber dimension of E.

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E),$$

defines the total Chern class.

(1) If E and $\bar{E}$ are complex bundles isomorphic over B, then

$$c(E) = c(\bar{E}).$$

If $E \xrightarrow{p} B$ and $f : \bar{B} \longrightarrow B$, then $f^*c(E) = c(f^*\bar{E})$.

(2) $c\left(E \oplus \bar{E}\right) = c(E)\,c(\bar{E})$, where the product denotes cup product in the cohomology ring of B, and E and $\bar{E}$ are complex bundles over B.

(3) Let $\Lambda \longrightarrow \mathbb{CP}^\infty$ be the canonical line bundle. Then

$$c_1(\Lambda) = i \in H^2(\mathbb{CP}^\infty)$$

as described above. Similarly, if $\lambda \longrightarrow S^2$ is the canonical line bundle over S^2, then $c_1(\lambda) = g \in H^2(S^2)$ is the generator.

It is known (the splitting principle) that given a complex bundle $E \xrightarrow{p} B$, then there exists a mapping $f : \bar{B} \longrightarrow B$ such that $f^\star$ injects the cohomology of B into the cohomology of $\bar{B}$ and $f^\star E$ is a direct sum of line bundles. Thus we can write

$$f^\star E \cong L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

whence

$$\begin{aligned} f^\star c(E) = c(f^\star E) &= c(L_1)\, c(L_2) \cdots c(L_n) \\ &= \prod_{k=1}^n \Big(1 + c_1(L_k)\Big). \end{aligned}$$

In this way, we see that the higher Chern classes can be expressed in terms of elementary symmetric functions of line bundles.

- **Example**

Let $E = \tau\, \mathbb{CP}_n =$ the tangent bundle to $\mathbb{CP}_n$. Explicitly,

$$\mathbb{CP}_n = S^{2n+1}/S^1$$

where S^1 is the unit complex numbers,

$$S^{2n+1} = \Big\{(z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} \big|\, |z_0|^2 + \cdots |z_n|^2 = 1\Big\}.$$

If $z = (z_0, z_1, \cdots, z_n)$ and $\lambda \in S^1$ then $\lambda z = \Big(\lambda z_0\, \lambda z_1, \cdots, \lambda z_n\Big)$. $E = \Big\{[u,v]\Big\}|\ \|u\| = 1,\ u \cdot v = 0,\ (u,v) \sim \Big(\lambda u,\, \lambda v\Big)\Big\}$. Here, $u, v \in S^{2n+1}$ and $[u,v]$ denotes the equivalence class of the pair (u,v) under the S^1-action.

Let $\Lambda_n \longrightarrow \mathbb{CP}_n$ denote the standard line bundle. Then,

$$\Lambda_n = \Big\{[u,\rho]\,|\, u \in S^{2n+1},\ \rho \in \mathbb{C},\ (u,\rho) \sim (\lambda u, \lambda\rho)\Big\}.$$

Let $E' = \Lambda_n \oplus \Lambda_n \oplus \cdots \oplus \Lambda_n\,(n+1\,\text{copies})$. Then,

$$E' = \Big\{[u,v]\,|\,(u,v) \in S^{2n+1} \times \mathbb{C}^{n+1},\ (u,v) \sim (\lambda u,\, \lambda v)\Big\}.$$

Hence, $E' \supset E$, and note that E' has the cross section $u \mapsto (u,v)$. Therefore, $E' \cong \tau\, \mathbb{CP}_n \oplus \epsilon$, where $\epsilon \to \mathbb{CP}^n$ denotes the trivial bundle in one complex dimension. We conclude that

$$c\Big(\tau\, \mathbb{CP}^n\Big) = c(E') = \Big(1 + c_1(\Lambda_n)\Big)^{n+1}.$$

Letting $\alpha_n = c_1(\Lambda_n)$ be the generator of $H^2(\mathbb{CP}_n)$, we have the formula

$$c\,(\tau\,(\mathbb{CP}^n)) = (1 + \alpha_n)^{n+1}.$$

14.0.2 Pontrjagin Classes

If $E \xrightarrow{p} B$ is a real vector bundle, then we get an associated complex vector bundle $\widehat{E} = E \otimes_{\mathbb{R}} \mathbb{C}$. Note that $\widehat{E}$ and its complex conjugate bundle are isomorphic, i.e. $\widehat{E}^* \cong \widehat{E}$. This implies that $2\, c_{2i+1}\left(\widehat{E}\right) = 0$. We define the i^{th} Pontrjagin class, $P_i(E)$, by the formula:

$$P_i(E) = (-1)^i\, c_{2i}\left(E \otimes_{\mathbb{R}} \mathbb{C}\right) \in H^{4i}(B),$$

and the total Pontrjagin class by the formula

$$P(E) = 1 + P_1(E) + \cdots + P_{[n/2]}(E),$$

where $[M]$ denotes the greatest integer in M. It then follows that

$$2\left(P(E \oplus E') - P(E)\,P(E')\right) = 0.$$

The following Lemma (whose proof we omit) is useful.

Lemma 14.1 *1.) Let ω be a complex vector bundle. Then $\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \omega^*$. (Here $\omega_{\mathbb{R}}$ denotes ω regarded as a real vector bundle.)*

2.) If $P_k = P_k(\omega_{\mathbb{R}})$, $c_k = c_k(\omega)$, then $1 - P_1 + P_2 - \cdots \pm P_n = (1 - c_1 + c_2 - c_3 + \cdots \pm c_n)\,(1 + c_1 + \cdots + c_n)$.

- Example.

$\tau = \tau\,\mathbb{CP}^n$, $c(\tau) = (1+a)^{n+1}$. $P_k = P_k(\tau_{\mathbb{R}})$. Then $1 - P_1 + P_2 - \cdots = (1-a)^{n+1}\,(1+a)^{n+1} = (1-a^2)^{n+1}$. Hence $1 + P_1 + \cdots + P_n = (1+a^2)^{n+1}$. Hence $P_k(\mathbb{CP}^n) = \binom{n+1}{k} a^{2k}$.

Now we apply the Pontrjagin classes to study manifolds. Let M^{4n} denote a smooth, compact $4n$ manifold without boundary. Let $M^{4n} \in H_{4n}(M;\mathbb{Z})$ denote the fundamental class of M^{4n}, and suppose that $i_1 + \cdots + i_r = n$, where $0 \le i_k \le n$. Let I denote the sequence $i_1, \cdots, i_r$ and define the Pontrjagin number $P_I[M^{4n}]$ by the formula

$$P_I\left[M^{4n}\right] = \left\langle P_{i_1} \cdots P_{i_r},\, M^{4n}\right\rangle$$

where the brackets denote the evaluation of the product $P_{i_1}, \cdots P_{i_r}$ on the fundamental class. For instance, from our last example, we see that

$$P_I\left[\mathbb{CP}^{2n}\right] = \binom{2n+1}{i_1}\cdots\binom{2n+1}{i_r}.$$

The following theorem is basic to the relationship of Pontrjagin classes and cobordism.

Theorem 14.1 *If the smooth manifold M^{4n} is the boundary of a smooth (4n+1)-manifold B^{4n+1}, $M^{4n} = \partial B^{4n+1}$ then all Pontrjagin numbers $P_I\ (M^{4n})$ vanish.*

Proof. Let μ_B denote the fundamental class in $H_{4n+1}(B, M)$. Then $\partial\mu_B = \mu_M$, where $\partial : H_{4n+1}(B, M) \to H_{4n}(M)$ is the homology boundary mapping. Furthermore, if $v \in H^{4n}(M)$ then $\langle v, \partial\mu_B\rangle = \langle\delta v, \mu_B\rangle$, where $\delta : H^{4n}(M) \to H^{4n+1}(B, M)$ is the coboundary map on cohomology. Now, we know that $\tau_B|M = \tau_M \oplus \epsilon$, hence $P_i(\tau_B|M) = P_i(\tau_M)$. It then follows directly from the exact sequence $H^{4n}(B) \xrightarrow{i^*} H^{4n}(M) \xrightarrow{\delta} H^{4n+1}(B, M)$ that $\delta(P_I) = 0$. Therefore,

$$\begin{aligned} P_I\ (M^{4n}) &= \langle P_I, \mu_M\rangle \\ &= \langle P_I, \partial\mu_B\rangle \\ &= \langle\delta(P_I), \mu_B\rangle \\ &= 0. \end{aligned}$$

This completes the proof ■

Thus we have shown that the $\mathbb{CP}^{2n}$ are not oriented boundaries. In fact, more is true. We can let Ω_n denote the oriented cobordism group of an n-dimensional smooth manifold. (Two oriented manifolds A^n and B^n are said to be cobordant if there exists an oriented $(n+1)$-manifold C^{n+1} such that $\partial C^{n+1} = A^n \cup (-B^n)$, where $-B^n$ denotes B^n with the reverse orientation. A manifold, A^n, is cobordant to $\varnothing$ if B^n can be taken to be empty. $\varnothing$ produces an inverse in cobordism classes since $\delta(A^n \times I) = A^n \cup (-A^n)$ and it is easy to see that the connected sum $A^n \cup B^n$ is cobordant to the connected sum $A^n \sharp B^n$. Thus, $A^n \sharp (-A^n)$ is cobordant to $\varnothing$.)

Ω_n is a ring with addition the operation of connected sum ($\sharp$) and multiplication the cartesian product. It is known that Ω_n is finite for $n \not\equiv 0 \pmod 4$ and that $\Omega_{4k} \otimes \mathbb{Q}$ has a basis

$$\left\{\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r} | I = i_1 i_2 \cdots i_r \text{ is a partition of } 4k\right\}.$$

See [M] for a proof of this result (due originally to René Thom).

We are now in a position to state and prove the fundamental theorem of Hirzebruch, connecting the signature of a $4k$-manifold with its Pontrjagin classes. The idea is to produce combinations of Pontrjagin classes that behave formally like the signature, and then use cobordism theory to check agreement on the relevant examples.

Recall the important properties of the signature, $\sigma\left(M^{4k}\right)$:

1) By definition, $\sigma\left(M^{4k}\right)$ is the signature of the quadratic form

$$\begin{aligned} H^{2k}(M) \times H^{2k}(M) &\longrightarrow \mathbb{Z} \\ a, b &\longmapsto \langle a \cup b, [M] \rangle . \end{aligned}$$

2) $\sigma(M_1 + M_2) = \sigma(M_1) + \sigma(M_2)$ where $M_1 + M_2 = M_1 \sharp M_2$, the connected sum.

3) If $M^{4k} = \partial N^{4k+1}$ then $\sigma\left(M^{4k}\right) = 0$.

Thus if M_1^{4k} is cobordant to M_2^{4k}, then $\sigma\left(M_1^{4k}\right) = \sigma\left(M_2^{4k}\right)$.

4) $\sigma\left(M_1^{4k} \times M_2^{4k}\right) = \sigma\left(M_1^{4k}\right) \sigma\left(M_2^{4k}\right)$.

Thus $\sigma : \Omega_* \longrightarrow \mathbb{Z}$ is a homomorphism from the cobordism ring to the integers. The Pontrjagin numbers already obey 2) and 3). We need to cook up property 4). For this, we need the concept of a multiplicative sequence: Let R be a commutative ring with unit, 1. Let $A^* = (A^0, A^1, A^2, \cdots)$ be a graded R-algebra. Let $A^\pi = \left\{a_0 + a_1 + a_2 + \cdots | a_i \in A^i\right\}$ be the associated formal power series ring. Let $K_i\ (x_1, x_2, \cdots, x_i)$ be a sequence of polynomials such that each K_n is homogeneous of degree n. Let

$$K : A^\pi \longrightarrow A^\pi \text{ via } K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + K_3(a_1, a_2, a_3) + \cdots .$$

We say that K is multiplicative if $K(ab) = K(a)K(b)$ for all $a, b \in A^\pi$.

Lemma 14.1 *Given a formal power series $f(t) = 1+\lambda_1 t + \lambda_2 t^2 + \cdots$, there exists a unique multiplicative sequence $\{K_n\}$ such that $K(1+t) = f(t)$.*

Proof.

For uniqueness, let $A^* = R[t_1, t_2, \cdots, t_n]$ and $\sigma = (1+t_1)(1+t_2)\cdots$ with $\sigma_1, \sigma_2, \cdots, \sigma_n$ the elementary symmetric functions so that

$$\sigma = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n.$$

Then

$$K(\sigma) = K(1+t_1)\,K(1+t_2)\cdots K(1+t_n) = f(t_1)\,f(t_2)\cdots f(t_n).$$

Thus $K(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is uniquely determined by $f(t)$. Since $\sigma_1, \sigma_2, \cdots, \sigma_n$ are algebraically independent, this proves uniqueness.

For existence, let $I = i_1 i_2 \cdots i_r$ be a partition of k and define

$$S_I\,(\sigma_1, \cdots, \sigma_n) = \sum t_1^{i_1} t_2^{i_2} \cdots t_r^{i_r}$$

where this sum means that we sum over all choices of r-subsets, thereby obtaining a symmetric function and hence a polynomial in the elementary symmetric functions $\sigma_1, \sigma_2, \cdots, \sigma_n$. These polynomials form a basis for the symmetric homogeneous polynomials of degree k in the variables $t_1, t_2, \cdots, t_n$. Thus, letting $\lambda_I = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}$, we can write

$$K_n\,(\sigma_1, \cdots, \cdots \sigma_n) = \sum_I \lambda_I S_I(\sigma_1, \cdots, \sigma_n)$$

where I ranges over all partitions of n. It follows that

$$S_I(ab) = \sum_{HJ=I} S_H(a)\,S_J(b),$$

where HJ denotes the partition obtained by juxtaposition. Hence

$$\begin{aligned} K(ab) &= \textstyle\sum_I \lambda_I\, S_I(ab) \\ &= \textstyle\sum_I \lambda_I \sum_{HJ=I} S_H(a)\, S_J(b) \\ &= \textstyle\sum_{H,J} \lambda_H\, S_H(a)\, \lambda_J\, S_J(b) \\ &= K(a)\,K(b). \end{aligned}$$

This completes the proof ■

Now let $\{K_n(x_1 \cdots x_n)\}$ be a multiplicative sequence of polynomials with rational coefficients. Let M^{4k} be a smooth compact oriented $4k$-manifold. Define the K-genus of M^{4k} by the formula

$$K[M^{4k}] = K_k[M^{4k}]\langle K_k(P_1, \cdots, P_k), [M^{4k}]\rangle,$$

where P_i denotes the i^{th} Pontrjagin class of τ_M. If $4 \nmid$ does not divide dim (M), define $K[M] = 0$.

Lemma 14.1 *If $\{K_n\}$ is any multiplicative sequence with rational coefficients, then the correspondence $M \mapsto K[M]$ defines a ring homomorphism $\Omega_* \longrightarrow \mathbb{Q}$ and hence an algebra homomorphism*

$$\Omega_* \otimes \mathbb{Q} \longrightarrow \mathbb{Q}.$$

Proof. We need only check the behavior on products. $M \times M'$ has total Pontrjagin class $P \times P'$ modulo elements of order 2. So $K((P \times P')) = K(P) \times K(P')$ and

$$\langle K(P) \times K(P'), \mu \times \mu'\rangle = (-1)^{mm'}\langle K(P), \mu\rangle\langle K(P'), \mu'\rangle.$$

Hence, $K[M \times M'] = K[M]\,K[M']$■

Now we can state and prove the Hirzebruch index theorem.

Theorem 14.1 (Hirzebruch) *Let $\{L_k\}$ be the multiplicative sequence of polynomials corresponding to $f(t) = \sqrt{t}/\tanh\left(\sqrt{t}\right)$. Then*

$$\sigma\left(M^{4k}\right) = L[M^{4k}].$$

Proof. By the quoted result on $\Omega_{4k} \otimes Q$, it suffices to check the theorem for $L_k[\mathbb{CP}^{2k}]$. Here $P = (1 + a^2)^{2k+1}$. Since $L(1 + a^2) = \sqrt{a^2}/\tanh\left(\sqrt{a^2}\right)$, $L(P) = (a/\tanh a)^{2k+1}$. Hence $L[\mathbb{CP}^{2k}] = \langle L(P), \mu\rangle$ equals the coefficient of a^{2k} in $\left(L\left(1 + a^2\right)\right)^{2k+1}$. We check this coefficient by residues. Let

$$u \tanh(z) = \left(e^z - e^{-z}\right) / \left(e^z + e^{-z}\right).$$

Then $du = (1 - u^2)dz$ whence

$$dz = \frac{du}{1-u^2} = (1 + u^2 + u^4 + \cdots)\,du.$$

$$\begin{aligned} \therefore\ L[\mathbb{CP}^{2k}] &= \tfrac{1}{2\pi i}\oint \tfrac{dz}{z^{2k+1}} \left(\tfrac{z}{\tanh z}\right)^{2k+1} \\ &= \tfrac{1}{2\pi i}\oint \left(\tfrac{dz}{\tanh z}\right)^{2k+1} \\ &= \tfrac{1}{2\pi i}\oint \tfrac{\left(1+u^2+u^4+\cdots\right)}{u^{2k+1}}\,du \\ &= 1. \end{aligned}$$

Hence, $L[\mathbb{CP}^{2k+1}] = 1 = \sigma\left(\mathbb{CP}^{2k}\right)$. This completes the proof ∎

Here are some useful facts about the series $\sqrt{t}/\tanh\left(\sqrt{t}\right)$:

$$\sqrt{t}/\tanh\left(\sqrt{t}\right) = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + - \cdots + \left(-1^{k-1}\,2^{2k}\,\frac{B_k\,t^k}{(2^k!)!} + \cdots\right)$$

where B_k is the k^{th} Bernoulli number. The first few L-polynomials are:

$$\begin{aligned} L_1 &= \tfrac{1}{3}P_1, \\ L_2 &= \tfrac{1}{45}\left(7P_2 - P_1^2\right), \\ L_3 &= \tfrac{1}{945}\left(62P_3 - 13P_1P_2 + 2\,P_1^3\right). \end{aligned}$$

14.1 Exotic Spheres

The example that we are about to discuss is not the first example of an exotic differentiable structure on a sphere, but it is diffeomorphic to that example. The first example is due to Milnor [2] and produces a non-standard differentiable structure on a sphere of dimension seven. The example we are about to discuss is due to Brieskorn [3].

The Brieskorn examples arise from studying algebraic varieties associated with polynomials of the form

$$f(z) = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$$

where $a_0, a_1, \cdots, a_n$ are positive integers and the z_i's are complex variables.

Let $V(f)$ denote the variety of f:

$$V(f) = \{z \in \mathbb{C}^{n+1} | f(z) = 0\}.$$

Let $K(f)$ denote the intersection of this variety with the unit sphere in $\mathbb{C}^{n+1}$:

$$\begin{aligned} K(f) &= V(f) \cap S^{2n+1} \\ &= \{z \in \mathbb{C}^{n+1} | f(z) = 0 \text{ and } |z| = |z_0|^2 + \cdots + |z_n|^2 = 1\}. \end{aligned}$$

It is not hard to check that $V(f)$ is a manifold away from $\vec{0} \in V(f)$ and that the intersection of $V(f)$ with S^{2n+1} is transversal. Hence, $K^{2n-1}(f)$ is a smooth manifold of dimension $2n - 1$.

Under these conditions, the manifolds $K^{2n-1}(f)$ are sometimes homeomorphic to spheres and sometimes cannot be diffeomorphic to standard spheres. A case in point is $K^7(f)$ for $f = z_o^3 + z_1^5 + z_2^2 + z_3^2 + z_4^2$. In general, let $\Sigma(a_0, a_1, \cdots, a_n)$ denote $K(f)$ for $f = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$. Thus we assert that $\Sigma(3,5,2,2,2) = \Sigma^7$ is an exotic sphere. We shall finish this chapter with a number of different points of view on this fact. Here are the facts that we will show:

1) $\Sigma^7 = \Sigma(3,5,2,2,2)$ is the boundary of a smooth 8-manifold of signature -8: $\Sigma^7 = \partial N^8$, $\sigma(N^8) = -8$.

2) Σ^7 is homeomorphic to a 7-dimensional sphere.

With these facts in hand, the exoticity of Σ^7 is proved as follows: An extra fact about the manifold N^8 is that it is connected and has vanishing homology except in dimension four. We can form the topological manifold $M^8 = N^8 \cup_\Sigma D^8$ where D^8, is a standard 8-ball. If M^8 is a smooth manifold, then $\sigma(M^8) = L[M^8]$. But $\widetilde{H}^*(M) = 0$ for $* \neq 4, 8$.

$$\begin{aligned} P_1 &\in H^4(M^8) \\ P_2 &\in H^8(M^8); \end{aligned}$$

and we have

$$\sigma(M^8) = L_2\left(M^8\right) = \frac{7}{45}\left[P_2(M^8) - P_1^2(M^8)\right];$$

$$-8 = \frac{7}{45}\left[P_2(M^8) - P_1^2(M^8)\right];$$

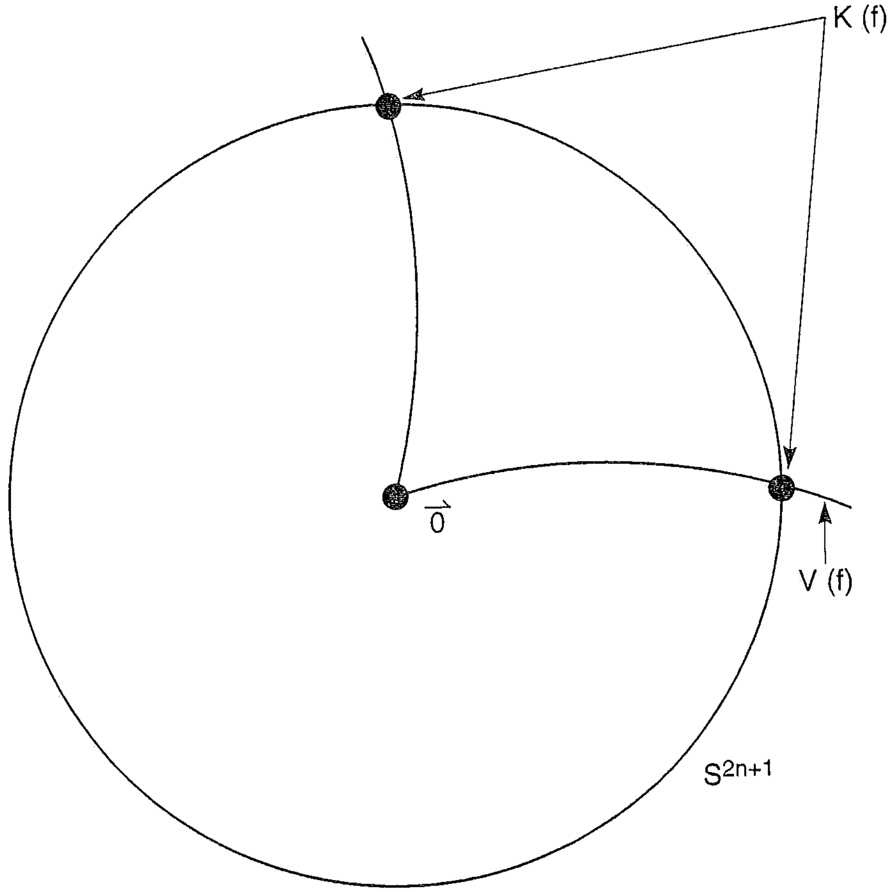

Figure 14.1:

$$-8 \cdot 45 = 7\left[P_2(M^8) - P_1^2(M^8)\right];$$

$$-2^3 \cdot 3^2 \cdot 5 = 7\left[P_2(M^8) - P_1^2(M^8)\right].$$

Since $(P_2(M^8) - P_1^2(M^8))$ is an integer and 7 does not divide $-2^3 \cdot 3^2 \cdot 5$, we conclude that M^8 does not have a differentiable structure. Since Σ diffeomorphic to S^7 would allow a differentiable structure on M^8, this shows that Σ is not diffeomorphic to S^7. Thus Σ is an exotic sphere.

For the record, Milnor's original example [2] of an exotic 7-sphere was constructed as follows: For each $(h, j) \in \mathbb{Z} \oplus \mathbb{Z}$ let $f_{h_j} : S^3 \longrightarrow \mathrm{SO}(4)$ be defined by the equation $f_{h_j}(u) \cdot v = u^h v u^j$ for $v \in \mathbb{R}^4$. Here we take quaternion multiplication on the right. Let ξ_{h_j} denote the 3-sphere bundle over S^4 determined by the map f_{h_j}. That is, with $S^4 = D^4_+ \cup_{S^3} D^4_-$, the quantity ξ_{h_j} is equivalent to $D^4_\pm \times S^3$ over $D^4_\pm$ and f_{h_j} provides the pasting data for gluing these two trivial bundles to form ξ_{h_j}. Let M^7_k denote the total space of the bundle ξ_{h_j} where $h + j = 1$ and $h - j = k$. Milnor shows that M^7_k is homeomorphic to S^7 for all k and that M^7_k is exotic when $k^2 \not\equiv 1 \pmod 7$. The argument involves the Pontrjagin classes of the bundle.

Now let us return to the Brieskorn manifolds and discuss some aspects of their structures. Consider $f(z) = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$ as a mapping $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$. It is easy to see that $f|\mathbb{C}^{n+1} - V(f) : \mathbb{C}^{n+1} - V(f) \longrightarrow \mathbb{C} - \{0\}$ is a fiber bundle, and that by taking the restriction to $E_\delta = f^{-1}(S^1_\delta) \xrightarrow{f} S^1_\delta$ where $S^1_\delta = \{z \in \mathbb{C} | |z| = \delta\}$ for δ small, we also get a fiber bundle and that $E_\delta \cap D^{2n+2} \longrightarrow S^1_\delta$ gives a fiber bundle with the boundary of each fiber diffeomorphic to $K(f)$. Milnor [9] generalized this fiber bundle structure to a bundle $\phi : S^{2n+1} - K(f) \longrightarrow S^1$, $\phi(z) = f(z)/|f(z)|$. In the case of $E_\delta \cap D^{2n+2} \longrightarrow S^1_\delta$ and $\phi : S^{2n+1} - K(f) \longrightarrow S^1$ are equivalent bundles by using the mapping

$$(z_0, z_1, \cdots, z_n) \longrightarrow \left(\rho^{1/a_0} z_0, \rho^{1/a_1} z_1, \cdots, \rho^{1/a_n} z_n\right)$$

for ρ real (choosing ρ so that the image point is on the sphere). In the general case (of f with an isolated singularity at the origin) Milnor uses a vector field to push the fibers of $E_\delta \cap D^{2n+2}$ out into the sphere.

A similar bit of geometric topology lets us see that $K\left(x^k + f(z)\right)$ is a k-fold branched cyclic cover of S^{2n+1} branched along $K(f)$. This sets the stage separate sets of variables) in terms of $K(f) \subset S^{2n+1}$ and $K(g) \subset S^{2m+1}$, where $f = f(z_0, \cdots, z_n), g = g(z_0, \cdots, z_m)$. Here, the idea is as follows. Suppose we are given maps $f : D^{2n+2} \longrightarrow D^2$ and $g : D^{2m+2} \longrightarrow D^2$ with singular fiber bundles elsewhere. Then we can form the pull-back $Z = \{(x,y) \in D^{2n+2} \times D^{2m+2} | f(x) = g(y)\}$:

$$\begin{array}{ccc} Z & \longrightarrow & D^{2m+2} \\ \downarrow & & \downarrow g \\ D^{2n+2} & \xrightarrow{f} & D^2. \end{array}$$

and $\partial Z \hookrightarrow \partial(D^{2m+2} \times D^{2n+2}) \cong S^{2(n+m)+3}$. Appropriate analysis shows that $\partial Z \subset S^{2(n+m)+3}$ is equivalent to $K(f+g) \subset S^{2(n+m)+3}$. See [4].

For example, in the case of $x^k + f(z)$, we have

$$\begin{array}{ccc} Z & \longrightarrow & D^2 \\ \downarrow & & \downarrow \\ D^{2n+2} & \xrightarrow{f} & D^2. \end{array}$$

with $g(x) = x^2$. Here it is easy to see that ∂Z is the k-fold cyclic branched covering of S^{2n+1} along $K(f)$. Note that this construction gives a canonical embedding in a sphere of two dimensions higher.

Thus, if $K^{n-2} \subset S^n$ then we have $K^n_a \longrightarrow S^n$ as branched cover, and $K^n_a \subset S^{n+2}$ where K^n_a denotes the a-fold cyclic branched cover of S^n.

In this way, we get an inductive definition of the Brieskorn manifolds as iterated branched coverings. $\Sigma(a_0, a_1)$ is a torus link of type (a_0, a_1) in S^3. For example $\Sigma(3,5) \subset S^3$ has diagram sketched in Figure 14.2.

Our Milnor sphere $\Sigma(3,5,2,2,2) \subset S^9$ is the result of three 2-fold branched coverings starting from the $(3,5)$ torus knot.

$$K_{3,5} \subset S^3 \longleftarrow K_{3,5,2} \subset S^5 \longleftarrow K_{3,5,2,2} \subset S^7.$$

These constructions give a clear view of the algebraic topology of the bounding manifolds. We shall only sketch these details here, referring the reader to [3], [4], and [5].

Given $K(f) \subset S^{2n+1}$ we have that $K(f) = \partial N(f)$ is the fiber of the Milnor fibration alluded above. The intersection form of middle dimension of

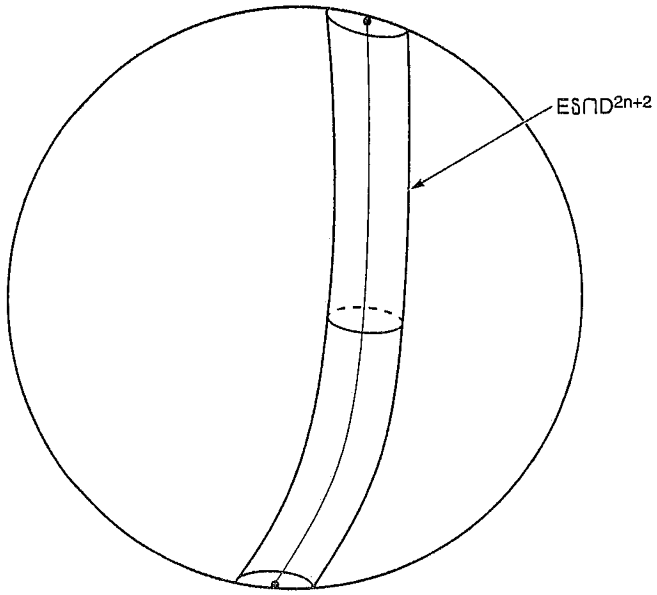

Figure 14.2:

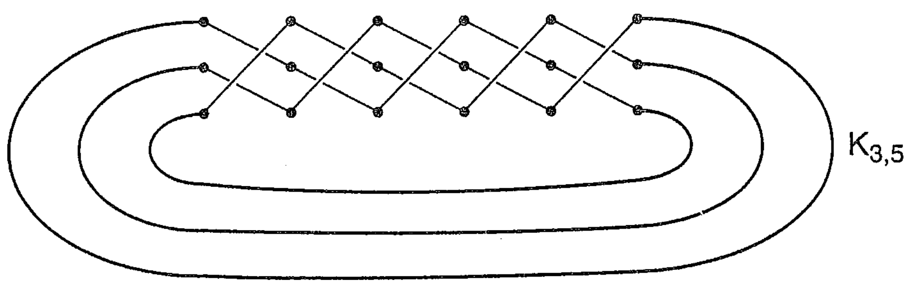

Figure 14.3:

Given $K(f) \subset S^{2n+1}$ we have that $K(f) = \partial N(f)$ is the fiber of the Milnor fibration alluded above. The intersection form of middle dimension of the homology of $N(f)$ is given by $\theta(f) \pm \theta(f)^{\mathsf{T}}$ where $\theta(f) : H_{n+1}(N(f)) \times H_{n+1}((f)) \longrightarrow \mathbb{Z}$ is the Seifert linking pairing obtained by the formula $\theta(f)(a,b) = lk(a^*, b)$ where lk denotes the linking number and a^* is the cycle in $S^{2n+1} - N(f)$ obtained by pushing a along a positive normal to $N(f)$ into the complement. One finds that $\theta(f+g) \cong \theta(f) \otimes \theta(g)$ and consequently it is easy to determine intersection forms for composites. In particular, one has $\theta(f + x^2) \cong \theta(f)$.

The construction we have discussed generalizes to a *tensor product construction* for $K^n \subset S^{n+2}$, $L^m \subset S^{m+2}$ (L is a fibered codimension two submanifold of S^{m+2} to $(K \otimes L)^{n+m+1} \subset S^{n+m+3}$). Thus, we can start with any knot $K \subset S^3$ and form

$$[K \otimes \Sigma(2,2,2)]^7 \subset S^9.$$

If θ is a Seifert pairing for K in S^3, then $K \otimes \Sigma(2,2,2) = \partial\mathcal{N}$, $\mathcal{N} \subset S^9$ with the same Seifert pairing. As a result, $\mathcal{N}$ has intersection pairing $\theta + \theta^{\mathsf{T}}$ and hence

$$\sigma(\mathcal{N}) = \sigma\left(\theta + \theta^{\mathsf{T}}\right) = \sigma(K),$$

the classical signature of the knot. As a consequence, many exotic spheres can be constructed directly in relation to knots and links in S^3.

The manifolds $K \otimes \Sigma(2,2,2,\cdots,2)$ (n2's) admit actions of the orthogonal group $O(n)$ with orbit space D^4 and fixed point set $K \subset S^3 = \partial D^4$. These are called link manifolds and are classified in [6]. We have discussed their relationship with global anomalies in [7].

It is also worth pointing out that the Brieskorn manifolds are tensor products of *empty knots* $[a] : \Sigma(a_0, a_1, \cdots, a_n) = [a_0] \otimes [a_1] \otimes \cdots, \otimes [a_n]$ where $[a] : S^1 \longrightarrow S^1$, $[a](\lambda) = \lambda^a$. The term $[a] : S^1 \longrightarrow S^1$ is a fibration corresponding to the empty knot $\psi \subset S^1$ (the empty set has dimension -1). By looking at the inverse image of a point in S^1 under $[a]$, we get the fiber consisting of discrete points (a in number), and hence the Seifert pairing of this empty knot. It has the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Finally, we should mention that so far we have only mentioned exotic spheres that are boundaries of parallelizable manifolds. There is a big class of exotic differentiable structures that do not bound in this way. Their properties require homotopy theory for detection. Such very exotic n-spheres are classified by $\pi_{n+k}(S^k)/\mathrm{Im}(J)$, where $\pi_{n+k}(S^k)$ denotes a stable homotopy group of the sphere S^k, and $\mathrm{Im}(J)$ denotes the image of the J- homomorphism:

$$J : \pi_n\left(\mathrm{SO}(k)\right) \longrightarrow \pi_{n+k}\left(S^k\right).$$

See [8] for more information on these matters.

Very exotic spheres may have some physical relevance, according to a conjecture by Witten. In reference [10] he postulates that gravitational instantons and/or solitons have the structure of very exotic spheres. Our knowledge of gravitational instantons and solitons is limited, but there is no doubt that deeper knowledge about very exotic spheres should shed light on these relationships. In addition to reference [10] where the role of very exotic spheres is detailed for the case of ten-dimensional supergravity theories, the interested reader may want to consult reference [11], [12], and [13] where the contributions to superstring theory of very exotic spheres are studied.

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