Exam1 With Solutions- Math 215 - Fall 2009

1. Let f_n denote the n-th Fibonacci number where $f_0 = f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n = 1, 2, \cdots$. Give a complete and careful proof by induction that

$$f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

for $n = 0, 1, 2, \cdots$.

Answer. Let P_n be the statement

$$f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

(where f_n denotes the n-th Fibonacci number, defined as in the problem statement.) We wish to prove that P_n is true for all $n = 0, 1, 2, \cdots$. For the base of the induction consider the case $n = 0$. Then P_0 states that $f_0^2 = f_0 f_1$, and since $f_0 = f_1 = 1$, this is true. For the induction step, assume that P_k is true. Then

$$f_0^2 + f_1^2 + f_2^2 + \cdots + f_{k+1}^2$$
$$= (f_0^2 + f_1^2 + f_2^2 + \cdots + f_k^2) + f_{k+1}^2$$
$$= f_k f_{k+1} + f_{k+1}^2$$

(by the induction assumption)

$$= f_{k+1}(f_k + f_{k+1})$$
$$= f_{k+1} f_{k+2}$$

(by the definition of the Fibonacci series). Thus we have proved that $P_k \Rightarrow P_{k+1}$. This completes the proof.

2. Recall that a proposition is said to be a tautology if it is always true. For example, $(\sim A) \vee A$ is a tautology. Use the following basic facts of logic

$$(A \Rightarrow B) = (\sim A) \vee B,$$
$$\sim\sim A = A, \;\; A \vee B = B \vee A,$$
$$A \vee (B \vee C) = (A \vee B) \vee C,$$

to show algebraically (without recourse to truth tables) that

$$(P \Rightarrow Q) \vee (Q \Rightarrow P)$$

is a tautology.

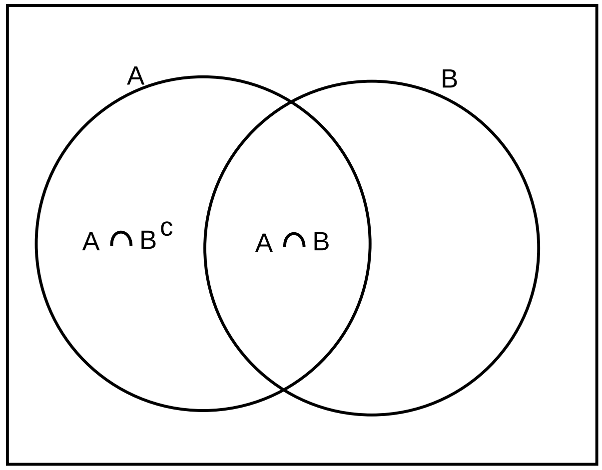
Answer.

$$(P \Rightarrow Q) \vee (Q \Rightarrow P)$$
$$= ((\sim P) \vee Q) \vee ((\sim Q) \vee P)$$
$$= (\sim P) \vee Q \vee (\sim Q) \vee P$$
$$= (\sim P) \vee P \vee (\sim Q) \vee Q$$
$$= ((\sim P) \vee P) \vee ((\sim Q) \vee Q)$$

(by using commutativity and associativity of $\vee$). Since $((\sim P) \vee P)$ and $((\sim Q) \vee Q)$ are each tautologies, and the "or" of two tautologies is a tautology, we conclude that $(P \Rightarrow Q) \vee (Q \Rightarrow P)$ is a tautology.

3. Prove by using Venn diagrams that for any two sets A and B,

$$(A \cap B^c) \cup (A \cap B) = A.$$



Answer. The diagram above illustrates that the union of the two sets $A \cap B^c$ and $A \cap B$ consists in all of the set A. This completes the proof using Venn diagrams.

4. Prove by contradiction that there does not exist a smallest positive real number.

Answer. Suppose that $r > 0$ is the smallest positive real number. Consider the fact that $0 < r/2 < r$. This is a contradiction and shows that there cannot be a smallest positive real number.

5. The following problem is due to the Reverend Charles Lutwidge Dodgson (27 January 1832 to 14 January 1898), also known as Lewis Caroll. He is the author of books on Symbolic Logic and also the books "Alice's Adventures in Wonderland" and "Through the Looking-Glass."

From the following three assertions we are to make whatever deductions are possible.

(i) *Nobody who really appreciates Beethoven fails to keep silence while the Moonlight Sonata is being played.*

(ii) *Guinea-pigs are hopelessly ignorant of music.*

(iii) *No one who is hopelessly ignorant of music ever keeps silence while the Moonlight Sonata is being played.*

These can be interpreted as statements about various sets. Let
G = the set of guinea-pigs.
H = the set of creatures that are hopelessly ignorant of music.
K = the set of creatures who keep silence while the Moonlight Sonata is being played.
R = the set of creatures that really appreciate Beethoven.

Rewrite each of **(i), (ii), (iii)** as a statement about sets in set theoretic notation. For example, statement (i) says that $R \subseteq K$. Use this rewrite to deduce that "Guinea pigs do not really appreciate Beethoven."

Answer. Note that a statement of the form "No X are not Y." is saying the same thing as "All X are Y." The first statement is equivalent to $R \subseteq K$. The second statement is equivalent to $G \subseteq H$. The third statement is equivalent to $H \subseteq K^c$. Thus $G \subseteq H \subseteq K^c$ so that $G \subseteq K^c$, while $R \subseteq K$. Therefore the intersection of R and G is empty. From this we conclude that "No Guinea pigs really appreciate Beethoven."