De Rahm Calculations and Cohomology ①

1° $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ (all functions are smooth)

$\Lambda^0 S^1 = \{f : S^1 \to \mathbb{R}\} \equiv \{f : \mathbb{R} \to \mathbb{R} \mid f(\theta + 2\pi) = f(\theta)\}$

$\Lambda^0 S^1 \xrightarrow{d} \Lambda^1 S^1$, $d(f) = \left(\frac{df}{d\theta}\right) d\theta$.

$\Lambda^2 S^1 = \{0\} \Rightarrow H^1_{DR}(S^1) = \Lambda^1(S^1)/d(\Lambda^0(S^1))$

$\longleftrightarrow \{f\,d\theta\}/\{\left(\frac{dg}{d\theta}\right)d\theta\} \longleftrightarrow \dfrac{\text{Periodic Functions}}{\text{Derivatives of Periodic Functions}}$

$\{f \mid f(\theta + 2\pi) = f(\theta)\} \xrightarrow{F} \mathbb{R}$

$F(f) = \int_0^{2\pi} f(\theta)\,d\theta$.

If $F(f) = 0$, define $g(\theta) = \int_0^{\theta} f(t)\,dt$.

Then $g(\theta + 2\pi) = \int_0^{\theta+2\pi} f(t)\,dt = \int_0^{2\pi} f(t)\,dt + \int_{2\pi}^{\theta+2\pi} f(t)\,dt$

$= 0 + \int_0^{\theta} f(s+2\pi)\,d(s+2\pi) = \int_0^{\theta} f(s+2\pi)\,ds$

$= \int_0^{\theta} f(s)\,ds = g(\theta)$.

Thus $F(f\,d\theta) = 0 \Rightarrow f\,d\theta = \frac{dg}{d\theta}\,d\theta$, so

$F : \Lambda^1(S^1)/d(\Lambda^0(S^1)) \xrightarrow{\cong} \mathbb{R}$

and we have $H^1_{DR}(S^1) \cong \mathbb{R}$.

2° Consider $T = S^1 \times S^1$.

$\Lambda^0(T) = \{f(\theta, \varphi) \mid f \text{ has period } 2\pi \text{ in both } \theta \text{ and } \varphi\}$

$\xrightarrow{d} \Lambda^1(T)$, $df = \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \varphi} d\varphi$

$\Lambda^1(T) \longrightarrow \Lambda^2(T)$

$d(a\,d\theta + b\,d\varphi) = \left(\frac{\partial b}{\partial \theta} - \frac{\partial a}{\partial \varphi}\right) d\theta \wedge d\varphi$.

More generally, you can think about
$M^n = \underbrace{S^1 \times S^1 \times S^1 \times \cdots \times S^1}_{n \text{ factors}}$

and functions that are n-fold periodic.

$$DR : \Lambda^K(M) \longrightarrow C^K(M; \mathbb{R})$$

$$DR(\omega)(\gamma) = \int_\gamma \omega$$

where γ is a K-chain on M.

$$\begin{array}{ccc} \Lambda^{K+1}(M) & \xrightarrow{DR} & C^{K+1}(M; \mathbb{R}) \\ \uparrow d & & \uparrow \delta \\ \Lambda^K(M) & \xrightarrow{DR} & C^K(M; \mathbb{R}) \end{array}$$

$$(\delta\, DR(\omega))(\beta) = DR(\omega)(\partial\beta)$$

$$= \int_{\partial\beta} \omega \equiv \int_\beta d\omega \quad \leftarrow \underline{\text{Stokes}} \ \underline{\text{Theorem}}$$

$$= DR(d\omega)(\beta)$$

$$\Longrightarrow \quad \delta(DR(\omega)) = DR(d\omega)$$

Thus DR is a map of cochain complexes.

One can prove that (for a general compact smooth n-manifold M^n)

$$H^*_{DR}(M^n) \xrightarrow[\cong]{DR_*} H^*(M; \mathbb{R}).$$

3° $\omega \in \Lambda^k(M^n)$, $\tau \in \Lambda^\ell(M^n)$,

$\omega \wedge \tau \in \Lambda^{k+\ell}(M^n)$.

Claim: $d(\omega \wedge \tau) = (d\omega)\wedge\tau + (-1)^k \omega \wedge d\tau$.

Thus, if $d\omega = 0$ and $d\tau = 0$,
then $d(\omega\wedge\tau) = 0$.

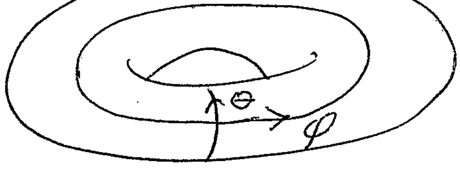
And if $\omega = d\lambda$ then
$$\omega\wedge\tau = (d\lambda)\wedge\tau = d(\lambda\wedge\tau)$$
And if $\tau = d\eta$ then
$$\omega\wedge\tau = \omega\wedge d\eta = d((-1)^k\omega\wedge\eta).$$

Conclusion: $\omega, \tau \longmapsto \omega\wedge\tau$
induces a well-defined mapping
$$H^k_{DR}(M) \times H^\ell_{DR}(M) \longrightarrow H^{k+\ell}_{DR}(M).$$
This is called the cup-product, an algebra structure on DeRahm Cohomology.

We will show, that infact the cup product is well-defined on general singular cohomology.

$$\underset{H^k}{\overset{\cap}{a}},\ \underset{H^\ell}{\overset{\cap}{b}} \longmapsto \underset{H^{k+\ell}}{\overset{\cap}{a \cup b}}$$

Example: $T = S^1 \times S^1$



$H^1_{DR}(T) \cong \mathbb{R} \oplus \mathbb{R}$ is generated by $[d\theta]$ and $[d\varphi]$

$\mathbb{R} \cong H^2_{DR}(T)$ is generated by $[d\theta \wedge d\varphi]$

and $[d\theta] \cup [d\varphi] = [d\theta \wedge d\varphi]$.

Example: $M^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$

periodic variables $\theta_1, \theta_2, \ldots, \theta_n$.

$H^k_{DR}(M^n)$ has rank $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

and is generated by

$\{ d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k} \mid i_1 < i_2 < \cdots < i_k \}$.

Thus, if $\alpha_i = [d\theta_i] \in H^1_{DR}(M^n)$ then

$\alpha_i \cup \alpha_i = 0$, $\alpha_i \cup \alpha_j = -\alpha_j \cup \alpha_i$

and so we see that

$H^*(\underbrace{S^1 \times \cdots \times S^1}_{n})$ is the

Grassmann algebra generated by $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

4° Cell Complex for M^n

S^1 : 

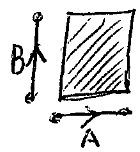
$\partial\sigma_1 = 0$
$(\& \; \partial\sigma_0 = 0)$

$M^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$: K-cells are of form

$$\sigma_1^{(i_1)} \times \cdots \times \sigma_1^{(i_K)}$$
$$i_1 < \cdots < i_K$$

and, using $\boxed{\partial(A \times B) = (\partial A) \times B + (-1)^K A \times \partial B}$
$(A \text{ a } K\text{-cell})$

$(N.B.\;\; \partial^2(\overset{K}{A} \times \overset{\ell}{B}) = \partial(\partial A \times B + (-1)^K A \times \partial B)$
$= \partial^2 A \times B + (-1)^{K-1}\partial A \times \partial B + (-1)^K \partial A \times \partial B$
$+ (-1)^K(-1)^K A \times \partial^2 B$
$= 0 + 0 + 0 = 0.)$

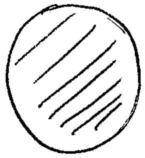


we get that
$$H_K(M^n) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\binom{n}{K} \text{ copies}}.$$

<u>Exercise</u>: Use this chain complex & cell complex to compute the integral cohomology of M^n.

5.° Homotopy and Cohomology

Higher homotopy groups $\pi_n(X,*)$:



$$\left[(D^n, S^{n-1}) \longrightarrow (X,*) \right]$$ homotopy classes of maps.

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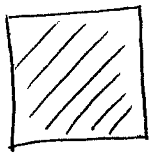

other models ...

$\pi_n(X,*)$ is abelian for $n \geq 2$.

$\exists$ spaces $K(\mathbb{Z},n)$ such that

$\pi_n(K(\mathbb{Z},n)) \cong \mathbb{Z}$

$\pi_\ell(K(\mathbb{Z},n)) \cong \{0\}$ $\ell \neq n$.

<u>Example</u>: $S^1 = K(\mathbb{Z},1)$

$\mathbb{CP}^\infty = K(\mathbb{Z},2)$ [$\mathbb{CP}^\infty$ = complex proj space]

<u>Fact</u>: $H^*(X;\mathbb{Z}) \cong [X, K(\mathbb{Z},n)]$.

The homotopy interpretation of cohomology creates a unified picture of all the basic functors in algebraic topology.

7° Cup Products in Singular Cohomology

Given an $n=p+q$ simplex $\Delta_n = \langle v_0 v_1 v_2 \ldots v_n \rangle$, define the front p-face $F_p\Delta_n = \langle v_0 v_1 \ldots v_p \rangle$ and the back q-face $B_q\Delta_n = \langle v_p v_{p+1} \ldots v_n \rangle$.

e.g. $4=1+3$

$$F_1\Delta_4 = \langle v_0 v_1 \rangle$$
$$B_3\Delta_4 = \langle v_1 v_2 v_3 v_4 \rangle.$$

With respect to this define the corresponding inclusion maps:

$$f_p : \Delta_p \hookrightarrow \Delta_n$$
$$b_q : \Delta_q \hookrightarrow \Delta_n.$$

Given cochains $a : C_p(X) \longrightarrow \mathbb{Z}$
$b : C_q(X) \longrightarrow \mathbb{Z}$

define $a \cup b : C_{p+q}(X) \longrightarrow \mathbb{Z}$ by

$a \cup b\,(g : \Delta_n \to X)$, $n=p+q$.

$$= a(\Delta_p \xhookrightarrow{f_p} \Delta_n \xrightarrow{g} X)\, b(\Delta_q \xhookrightarrow{b_q} \Delta_n \to X).$$

Lemma. $\boxed{\delta(a \cup b) = \delta(a) \cup b + (-1)^p a \cup \delta(b).}$

It then follows that

$$C^* \times C^\# \xrightarrow{\cup} C^{*+\#}$$

induces a well-defined map

$$H^p(X) \times H^q(X) \xrightarrow{\cup} H^{p+q}(X).$$

This is the general definition of the cup product in singular cohomology.