



[Easy Graph Software](#)

Create Business Graphs Fast. See Examples. Free Download!
www.SmartDraw.com

[Graphing Linear Equations](#)

Graph any linear equation fast with Algebrator's step-by step solutions
www.Algebra-Help.com/Graphing

[Trigg Industries](#)

Overheight Vehicle Detection and Warning Systems
www.triggindustries.com/

[DataArt NETChart](#)

.NET Charts: Bar, Pie, Line, Area, Scatter, Marker graphs. Free trial.
www.dataart.com

[Manifesto](#) [Search](#) [Bookstore](#) [Contents](#) [Glossary](#) [CTK Updates](#) [Mail](#) [Recommend](#)

Crossing Number of a Graph

A store of hand-picked math books and of everything else
Stop and shop.
<http://astore.amazon.com>

Google

 Web CTK

[Best sites for teachers](#)
[Sites for teachers](#)
[Sites for parents](#)
[Terms of use](#)
[Awards](#)

[Interactive Activities](#)
[CTK Exchange](#)
[CTK Insights - a blog](#)

[Games & Puzzles](#)
[What Is What](#)
[Arithmetic/Algebra](#)
[Geometry](#)
[Probability](#)
[Outline Mathematics](#)
[Make an Identity](#)
[Book Reviews](#)
[Eye Opener](#)
[Analog Gadgets](#)
[Inventor's Paradox](#)
[Did you know?...](#)
[Proofs](#)
[Math as Language](#)
[Things Impossible](#)
[Visual Illusions](#)
[My Logo](#)
[Math Poll](#)
[Cut The Knot!](#)
[MSET99 Talk](#)
[Other Math sites](#)
[Front Page](#)
[Movie shortcuts](#)
[Personal info](#)
[Reciprocal links](#)
[Privacy Policy](#)

[Guest book](#)
[News sites](#)

[Recommend this site](#)

[Games to relax](#)

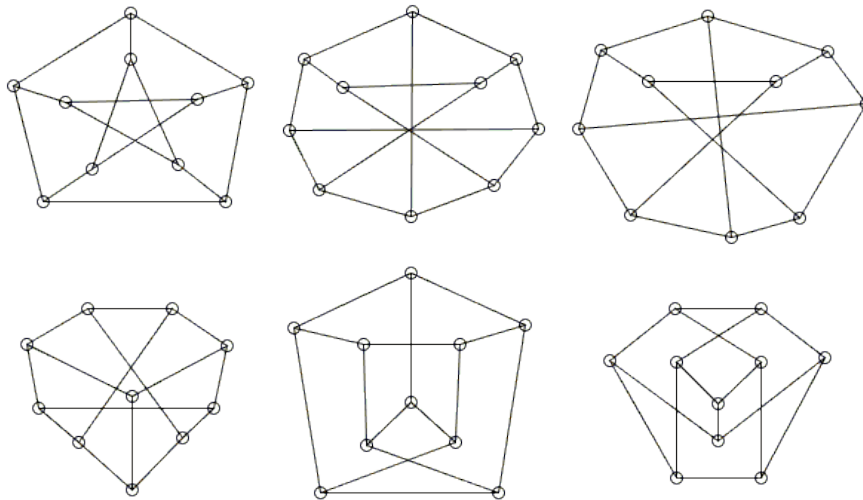
[Best sites for teachers](#)
[Sites for teachers](#)
[Sites for parents](#)

[Advertise on this site](#)
Powered By AdBrite

[Your Ad Here](#)

[Press collection](#)
[Education & Parenting](#)

A [graph](#) which is essentially an algebraic structure of nodes and edges is commonly presented geometrically by diagrams depicting nodes as dots and edges as curves or straight line segments that connect the dots. Such graphic representations are far from being unique. Below you can see several such *avatars* of the [Petersen graph](#) [[Balakrishnan](#), iv]:



An obvious fact about these diagrams is that sometimes the edges cross at the points that do not belong to the graph. In other words, in diagrammatic incarnations, the edges may occasionally meet at the points that are not the nodes of the graph. For [planar graphs](#), there is always a representation that avoids such crossings at all. But not all graphs are planar. And for the latter kind, an important characteristics is their *crossing number*, the minimal number of edge crossings among all possible diagrammatic representations of the graph in a plane. Note that, for the sake of the crossings count, avatars like the second in the upper row above are not allowed. What disqualifies it is the presence of a crossing where more than two edges meet. However, it is always possible either by edge distortion or by shifting the nodes to change the diagram to a permissible one (the right most in the upper row). The crossing number of a graph is often denoted as **k** or **cr**.

Among the six incarnations of the Petersen graph, the middle one in the bottom row exhibits just 2 crossings, fewer than any other in the collection. In fact, 2 is the crossing number of the Petersen graph. Try as you may, it is impossible to diagram the Petersen graph with one or zero crossings. The fact begs a proof. In the proof below, I'll use an argument suggested by Nathan Bowler in one of the online discussions at the [CTK Exchange](#).

Necessary for the proof is the notion of *girth*. The *girth* of a graph is the length of the shortest [cycle](#) the graph contains. (Here I assume that the graph does not have parallel edges, i.e. edges of multiplicity higher than 1, nor the [loops](#).) I shall use the symbol **c** to denote the girth. Always $c \geq 3$. For the Petersen graph, $c = 5$. (Looking at the leftmost avatar in the upper row above, the outer pentagon and the inner star are 5-cycles. Other cycles are bound to contain an even number of "bridges". But if two bridges are adjacent at one end they are at least two edges apart at the other.)

For every avatar, let's consider a [subgraph](#) obtained by removing one of the edges at every crossing present. This operation is not going to increase the graph's girth. Furthermore, the operation leaves a graph with no crossings, i.e., a planar graph. For the planar graphs, we have Euler's polyhedral formula:

$$f - e + v = 2,$$

where **f**, **e**, and **v**, are correspondingly the number of faces (countries), edges, and vertices of a graph. For a planar graph with girth **c**,

Search New and Used
amazon.com.

Search:

Books

Keywords:

Cut the knot:
Learn to enjoy!

I Recommend

Sacred Mathematic...
Princeton Unive...
\$23.45

A truly cultural event: besides a multitude of ...

Impossible?: Surpri...
Princeton Unive...
\$18.45

An exceptionally well written discussion of 1...

[amazon.com](#)

.net Charting

FREE DEVELOPER VERSION
DOWNLOAD NOW

.netCHARTING
 helps you look
 your best.

.net Charting

www.dotnetcharting.com
 Ads by Google

$$fc \leq 2e.$$

The two formulas together give

$$(1) \quad 2e - ce + cv \geq 2c.$$

If E is the original number of edges, then $e = E - k$ and inequality (1) implies that

$$(2) \quad k \geq E - (v - 2) \cdot c / (c - 2).$$

For the Petersen graph, $E = 15$, $v = 10$, $c = 5$, so that (2) gives

$$k \geq 5/3,$$

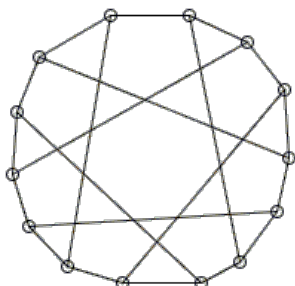
or since k is an integer,

$$k \geq 2.$$

Finally, since in one of the above diagram the number of crossings is exactly 2, we conclude that, for the Petersen graph,

$$k = 2.$$

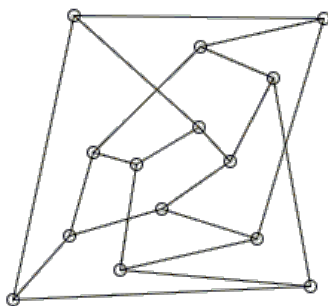
Another example of using (2) is served by the *Heawood graph*:



For the graph, $c = 6$ (this needs a [proof](#)), $E = 21$, $v = 14$. Substituting these into (2) gives

$$k \geq 21 - 12 \cdot 6 / 4 = 3.$$

But it is possible to depict the Heawood graph with just three crossings:



implying that its crossing number is exactly 3.

For the [prototypical](#) non-planar graphs $K_{3,3}$ and K_5 (2) also proves useful. Indeed, for $K_{3,3}$, $c = 4$, $E = 9$, $v = 6$, so that $k \geq 1$. For K_5 , $c = 3$, $E = 10$, $v = 5$, so that again $k \geq 1$. In this manner, via (2), we get an additional proof of the non-planarity of $K_{3,3}$ and K_5 .

Remarks

1. Since $c \geq 3$, (2) implies that for planar graphs

[Get Widget](#) Privacy

Latest on [CTK Exchange](#)

[Math](#)
 Posted by Laura
 2 messages
 06:56 AM, Apr-15-08

[Constructing a triangle instructions](#)
 Posted by Gerald B.
 3 messages
 01:32 PM, May-20-08

[drawing puzzle](#)
 Posted by martin gran
 31 messages
 06:53 PM, May-09-08

[Question on the Limits on Paper F ...](#)
 Posted by Cliff P
 4 messages
 08:48 PM, May-23-08

[Mistake on the page \(an aside, Be ...](#)
 Posted by Max
 4 messages
 10:28 AM, Feb-28-08

[Need details on a part of Proof o ...](#)
 Posted by Manuel S.
 2 messages
 05:24 PM, May-16-08

[Josephus Flavius \(correction\)](#)
 Posted by David Turner
 1 messages
 09:42 AM, May-14-08

Deal Of 1

Lightning

$$E \leq 3v - 6.$$

For K_5 we would have, $10 \leq 9$, which again shows that the graph is non-planar.

1. Richard Guy showed (1972) that for complete graphs K_n ,

$$k(K_n) \leq \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor \lfloor (n-2)/2 \rfloor \lfloor (n-3)/2 \rfloor / 4,$$

with equality proven for small n . To the best of my knowledge, his conjecture that the equality always holds is still unproven.

Reference

1. V. K. Balakrishnan, [Graph Theory](#), Schaum's Outlines, 1997
2. N. Hartsfield, G. Ringel [Pearls in Graph Theory: A Comprehensive Introduction](#), Dover, 1994
3. R. J. Trudeau, [Introduction to Graph Theory](#), Dover, NY, 1993.

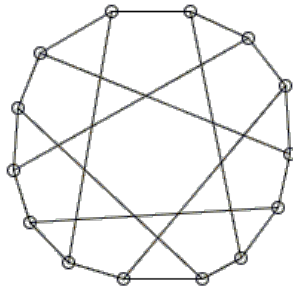


- [Graphs](#)
 - [3 Utilities Puzzle](#)
 - [Crossing Number of a Graph](#)
 - [Regular Polyhedra](#)
 - [The Affirmative Action Problem](#)
 - [The Two Men of Tibet Problem](#)
- [Puzzles on graphs](#)
 - [Three Glass Puzzle \(example\)](#)
 - [Three Glass Puzzle in Barycentric Coordinates \(example\)](#)
 - [Sierpinski Gasket and Tower of Hanoi \(example\)](#)
 - [Sam Loyd's Fifteen](#)
 - [Sliders](#)
 - [Lucky 7](#)
 - [Happy 8](#)
 - [Blithe 12](#)
 - [Slider puzzles](#)
- [Permutations](#)
 - [Various ways to define a permutation](#)
 - [Counting and listing all permutations](#)
- [Transpositions](#)
 - [A Shuttle Puzzle](#)
- [Groups of Permutations](#)
 - [Group multiplication of permutations](#)



[\[Mail\]](#) [\[Front Page\]](#) [\[Contents\]](#) [\[Store\]](#) [\[Geometry\]](#)

Copyright © 1996-2008 [Alexander Bogomolny](#)



Referring to the diagram, the edges of the Heawood graph divide in an obvious way into long and short edges. Clearly there is no such cycle composed entirely of short edges, or with just 1 long edge. If there were 2 long edges, they could not be adjacent and so would both contribute +5 to the cycle. The remaining edges (at most 3) could neither complete this to a full loop (of length 14) or reduce it to 0. There could not be more than 2 long edges, as then some two would have to be adjacent.



[|Mail|](#) [|Front Page|](#) [|Contents|](#) [|Store|](#) [|Geometry|](#)

Copyright © 1996-2008 [Alexander Bogomolny](#)

28903082 PAGE PROTECTED BY **COPYSCAPE** DO NOT COPY



CITE THIS PAGE AS:

A. Bogomolny, [Crossing Number of a Graph](#) from *Interactive Mathematics Miscellany and Puzzles*
http://www.cut-the-knot.org/do_you_know/CrossingNumber.shtml, Accessed 27 May 2008

<p>Free Family Tree Program Create and Print Your Family Tree Easy and Free! www.myheritage.com</p>	<p>Sutherland Crossing Discount condo Rentals & Resales Rent and sell timeshares By Owners www.MyResortNetwork.com</p>	<p>Freeware Router Monitor PRTG monitors & graphs router traffic. Freeware. Download. www.Paessler.com/prtg</p>	<p>Winterpark Hotels Low Rate Guarantee, Plus Pictures, Reviews & More! www.BookIt.com</p>
---	--	--	--

Ads by Google