A GROUP TO CLASSIFY KNOTS

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Simon [2] has shown that we can effectively associate to each (tame) knot K in S^3 a finitely presented *classifying group* $CG_R(K)$ so that two knots K and K' are equivalent if and only if $CG_R(K)$ and $CG_R(K')$ are isomorphic. $CG_R(K)$ is defined in terms of certain cables of $K \# R$, R being some fixed reference knot, and Simon's result depends on several geometrical facts, including Waldhausen's results [3] on sufficiently large 3-manifolds. In this note we produce a simpler classifying group which can be derived from Waldhausen's results in a purely algebraic fashion.

THEOREM. *Let $G = (g_i \mid r_j)$ be a presentation for the fundamental group of a knot K, and let λ denote a longitudinal, and μ a corresponding meridional element, properly oriented* (see Fig. 1). *Then the group $G_{\lambda,\mu}$ with presentation*

$$(g_i, a, b \mid r_j, a^5 = 1, \quad \lambda^{-1} a\lambda = a^2, \quad b^7 = 1, \quad \mu^{-1} b\mu = b^2, \quad [a, b] = 1)$$

is a classifying group for K.

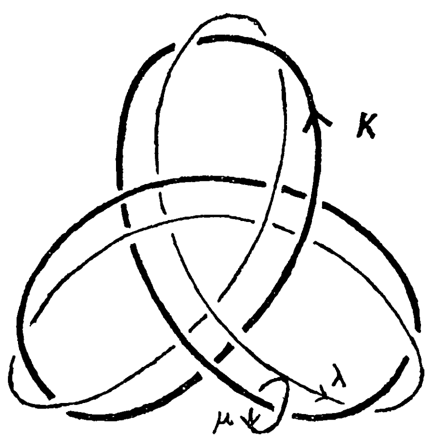

Fig. 1. A knot K and its peripheral elements λ, μ.

Notes. With base point on the boundary of some tubular neighbourhood T of K, λ is represented by a path on ∂T homologous to K in T and not homologically linking K, and μ by a path on ∂T spanning a disc in T and linking K just once in the positive sense. Since the orientations of space and knot are taken into account, our result is in fact slightly stronger than Simon's—if we did not wish to consider orientations we could replace a^2 and b^2 by a and b in the above relations.

Proof. First suppose that K is non-trivial. Then [1] λ and μ generate a subgroup

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of G which is free abelian of rank two. Hence $G_{\lambda,\mu}$ is an amalgamated free product $G_{*P}H$ of G with the group

$$H = (\lambda, \mu, a, b \mid a^5 = 1, \quad \lambda^{-1} a\lambda = a^2, \quad b^7 = 1, \quad \mu^{-1} b\mu = b^2, \quad [a, b] = [\lambda, \mu] = 1),$$

the amalgamated subgroup P being generated by λ and μ. Since G is torsion-free **[1]**, it follows that the only elements of finite order in $G_{\lambda,\mu}$ are conjugates of powers of ab.

Note also that every automorphism of the cyclic subgroup generated by ab extends to an automorphism of $G_{\lambda,\mu}$.

Let $\langle\ldots\rangle$ denote normal closure. We then have

$$G_\lambda = G_{\lambda,\mu}/\langle b\rangle \cong G_{*L} X,$$

where $X = (\lambda, a \mid a^5 = 1, \lambda^{-1} a\lambda = a^2)$, and the amalgamated subgroup L is infinite cyclic generated by λ.

Similarly,

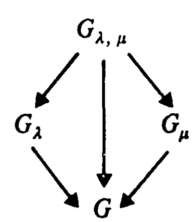
$$G_\mu = G_{\lambda,\mu}/\langle a\rangle \cong G_{*M} Y,$$

where $Y = (\mu, b \mid b^7 = 1, \mu^{-1} b\mu = b^2)$, and M is infinite cyclic generated by μ.

Also,

$$G_{\lambda,\mu}/\langle ab\rangle \cong G_\lambda/\langle a\rangle \cong G_\mu/\langle b\rangle \cong G,$$

and we have the following consistent diagram Δ of quotient homomorphisms



Now consider $G_\lambda \cong G_{*L} X$, and let A be the subgroup of G_λ generated by a. Take the powers of a as coset representatives for L in X, and similarly select coset representatives $1, g, g', \ldots$ for L in G. Then the word theorem for amalgamated free products tells us that coset representatives for L in $G_{*L} X$ are the elements expressible as products of non-trivial coset representatives of L alternately in G and in X, and that the expression for any such element in this form is unique.

We deduce that X is the normaliser of A in G_λ. For the relations certainly show X is part of this normaliser, and if the normaliser had any other element it would therefore have one of the form $g_0 a^i g_1 a^j \ldots a^k g_n$, the g_t being non-trivial coset representatives of L in G, and the powers of a being $\neq 1$. But the equation

$$ag_0 a^i g_1 a^j \ldots a^k g_n = g_0 a^i g_1 a^j \ldots a^k g_n a^s$$

defining the action of this on a is then prohibited, since both sides are in normal form.

The cyclic subgroup L of G is therefore determined as the image of the normaliser of A under the quotient homomorphism from G_λ onto G. This group has two generators λ and λ^{-1}, and λ can be defined as the unique one of these that is the image of some x in G_λ with $x^{-1}ax = a^2$. (We cannot have both $x^{-1}ax = a^2$ and $xax^{-1} = a^2$, for these relations imply $a^3 = 1$.)

Consideration of G_μ gives an analogous characterisation of μ.

Now suppose that $G_{\lambda,\mu}$ and $G'_{\lambda',\mu'}$ are the groups associated with two knots K and K'.

First recall that K is trivial if and only if G is infinite cyclic [**1**]. It is then clear that if F denotes the set of elements of finite order in $G_{\lambda,\mu}$, we have $G_{\lambda,\mu}/\langle F\rangle \cong G$ in this case also. It follows that if $G_{\lambda,\mu}$ is isomorphic to $G'_{\lambda',\mu'}$, and K is trivial, then K' is trivial.

Suppose, then, that K and K' are both non-trivial, and that $\phi : G_{\lambda,\mu} \to G'_{\lambda',\mu'}$ is an isomorphism. Since the elements of finite order in $G_{\lambda,\mu}$ are conjugates of powers of ab, and similarly for $G'_{\lambda',\mu'}$, we may suppose (by composing ϕ with a suitable automorphism of $G'_{\lambda',\mu'}$) that $\phi(ab) = a'b'$. Then ϕ induces isomorphisms $\phi_l : G_\lambda \to G'_{\lambda'}$, $\phi_m : G_\mu \to G'_{\mu'}$, and $\psi : G \to G'$, which are compatible with the diagrams Δ and Δ' of quotient homomorphisms, and such that $\phi_l(a) = a'$ and $\phi_m(b) = b'$. It then follows from our characterisations of λ and μ in G that $\psi(\lambda) = \lambda'$ and $\psi(\mu) = \mu'$.

It only remains to appeal to Waldhausen's theorem [**3**; Corollary 6.5] to conclude that ψ is induced by a homeomorphism from $S^3 \setminus \mathring{T}$ onto $S^3 \setminus \mathring{T}'$, T and T' being appropriate tubular neighbourhoods of K and K'. Since this homeomorphism respects our longitudinal and meridional elements, we can extend it so as to take T onto T', and so obtain a homeomorphism of S^3 taking K onto K', preserving all orientations.

References

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