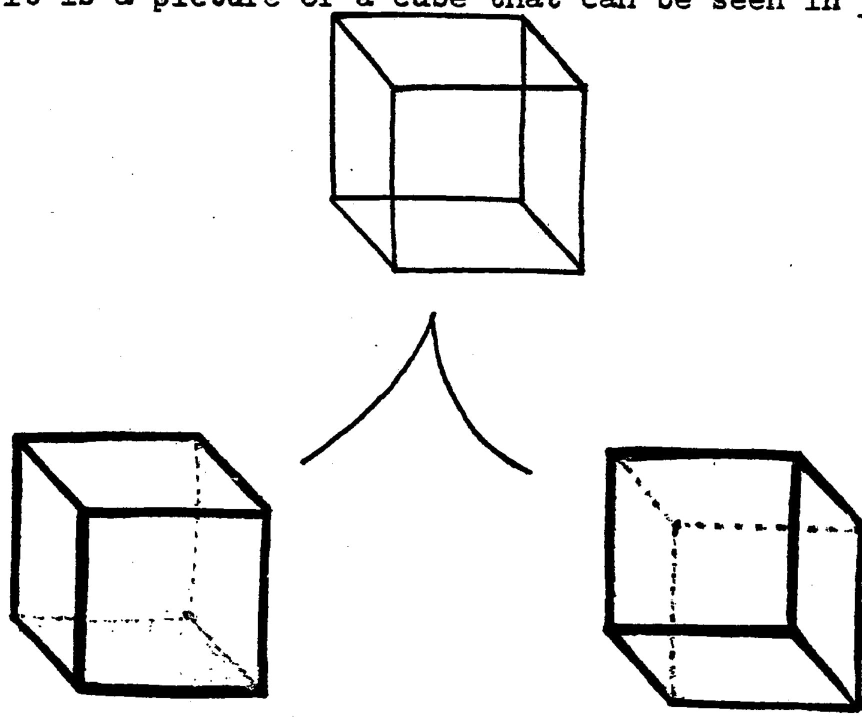
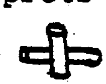
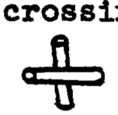
III. Complex Numbers and Optical Illusions.

You are probably familiar with the so-called Necker cube illusion: It is a picture of a cube that can be seen in two ways.



The illusion, and the way we tend to oscillate back and forth between the two views, is generated by the ambiguity of  which the mind interprets as a crossing of type  or of type . That is, like  or  .

Of course both cubic views are just in your mind, and certainly the difference between the two views is purely imaginary (a matter of image-ination)!

I like to think of $+\sqrt{-1}$ and $-\sqrt{-1}$ as being something like the two views of the Necker cube. They are both our attempts to "jump out" of the oscillating paradox presented by $x = -1/x$. How, you may ask, do you arrive at two views from the paradox?

Well, let's see (I'd better think fast.), the paradox arises by trying to solve $x = -1/x$ by iteration. We start with -1 and get $+1$, then -1, then $+1, \ldots$

$$-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, \ldots$$

But we could have started with $+1$ and then

$$+1, -1, +1, -1, +1, -1, +1, -1, +1, -1, \ldots$$

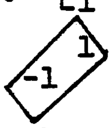
If you and your friend had each started at the same time, one with $+1$, the other with -1, you'd find yourselves chanting opposite numbers at the same time! Each taking a different view of $\sqrt{-1}$.

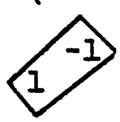
Let's put this another way. Suppose I start intoning:

PLUS, MINUS, PLUS, MINUS, PLUS, MINUS,...

and you listen to me. After a while it will start sounding <u>PLUS</u> MINUS, <u>PLUS</u> MINUS,... and then it will switch to <u>MINUS</u> PLUS, <u>MINUS</u> PLUS, <u>MINUS</u> PLUS,... <u>I'm</u> not doing that! You're doing it. The difference is in your imagination, and you get two views (soundings really) of my chanting in a way perfectly analogous to the Necker cube illusion!

Thus we might try to say that $\sqrt{-1}$ is "really" the pair [+1,-1] (or is it [-1,+1]?) where this denotes the <u>PLUS</u> MINUS take on the sequence ... PLUS MINUS PLUS MINUS...

It is actually possible to make mathematics out of these ideas. The algebraic version is called <u>matrix algebra</u>. In matrix algebra we represent $\sqrt{-1}$ by an array $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $-\sqrt{-1}$ by an array $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (I'll tell you about the zeroes in a moment). Notice how 

and 

are the basic ingredients in the array, just as we have wanted.

The zeroes are locations for the <u>real part</u> (which does not oscillate). In general a complex number $a + b\sqrt{-1}$ is represented by the array $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Thus $3 + 4\sqrt{-1}$ corresponds to $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$.

In this system the array $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ corresponds to $1 + 0\sqrt{-1} = 1$, the number one. The next exercise shows you how to multiply these matrices to make things work out right. For more information read the chapter on matrices in the text.

<u>Optional Exercise 5</u>. Define the product of two matrices by the formula

$$\begin{bmatrix} x & z \\ w & y \end{bmatrix}\begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} xA + zD & xC + zB \\ wA + yD & wC + yB \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 2 & 1\cdot 1 + 2\cdot 1 \\ 3\cdot 1 + 4\cdot 2 & 3\cdot 1 + 4\cdot 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 11 & 7 \end{bmatrix}$$

Show: a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} -A & -C \\ -D & -B \end{bmatrix}.$

b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

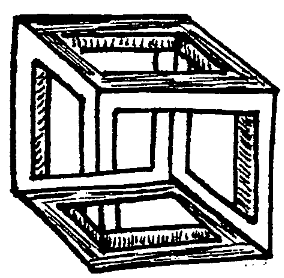
(This corresponds to $(\sqrt{-1})^2 = -1$.)

c) $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac - bd & ad + bc \\ ad + bc & ac - bd \end{bmatrix}.$

This corresponds to $(a + bi)(c + di) = (ad - bc) + (ad + bc)i$.

__Exercise 6__. Give as many examples as you can of ambiguous situations (like the Necker cube) that have multiple interpretations. (Look at other optical illusions, word games, things you see and hear,... perhaps after awhile and some imagining you'll begin to wonder what doesn't have a multiple interpretation!)

__Exercise 7__. __Discuss:__



__Exercise 8__. __Discuss:__

$$-1 = (\sqrt{-1})^2 = (\sqrt{-1})(\sqrt{-1})$$
$$= \sqrt{(-1)(-1)}$$
$$= \sqrt{+1}$$
$$\therefore \quad -1 = +1$$