Euler Formula Facts

1° $\Gamma \subset \mathbb{R}^2$, Γ a connected graph

$\Rightarrow$ $\boxed{v - e + f = 2.}$

2° $\Gamma \subset S_g$, S_g an orientable surface of genus g, Γ a <u>tight</u> embedding (all regions are disks after removal of the boundary of the region), Γ connected.

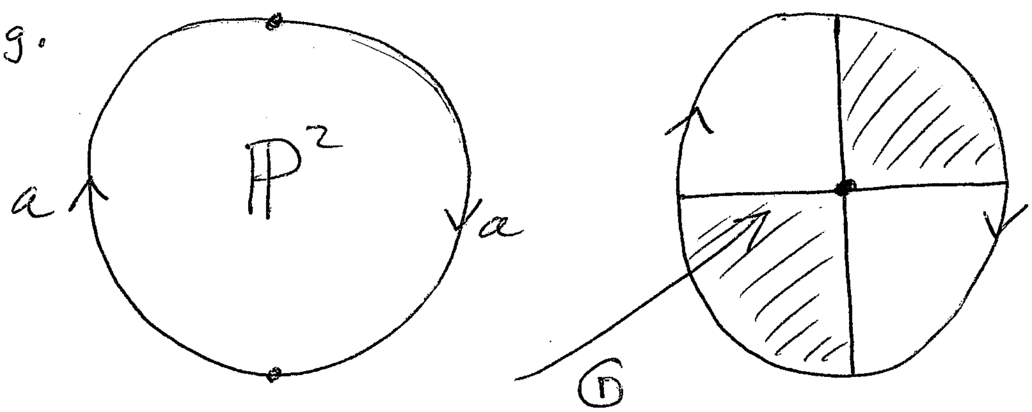
$\Rightarrow$ $\boxed{v - e + f = 2 - 2g}$.

3° $\Gamma \subset \mathbb{P}^2$ ($\mathbb{P}^2$ = the projective plane)

Γ a tight graph embedding, Γ connected.

$\Rightarrow$ $\boxed{v - e + f = 1}$.

e.g.



$v = 1,\ e = 2,\ f = 2$

$v - e + f = 1$

4° $\Gamma \subset S_{g,r}$ = sphere with g handles and "r cross-caps".

$= \underbrace{T \# T \# \cdots \# T}_{g} \# \underbrace{\mathbb{P} \# \mathbb{P} \# \cdots \# \mathbb{P}}_{r}$

Γ tight, connected.

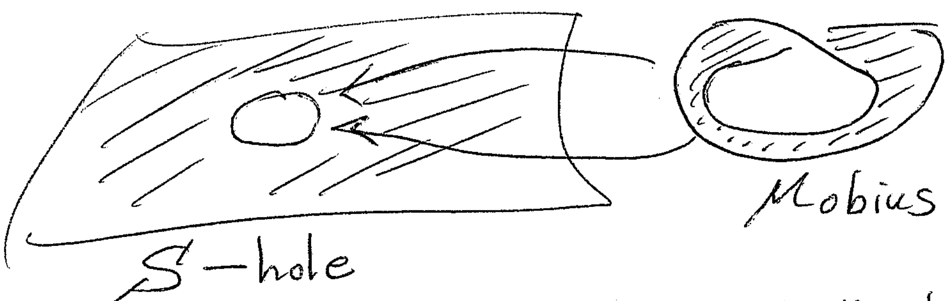
$\Rightarrow$ $\boxed{v - e + f = 2 - 2g - r}$

Note that 4.° is compatible with the fact that

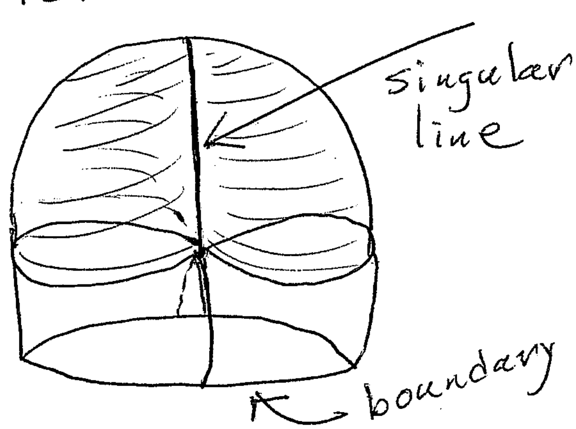
$$\mathbb{P} \# \mathbb{P} \# \mathbb{P} \cong \text{Klein} \# \mathbb{P} \cong T \# \mathbb{P}$$

since $2 - 2 \cdot 1 - 1 = 2 - 3 \cdot 1$.

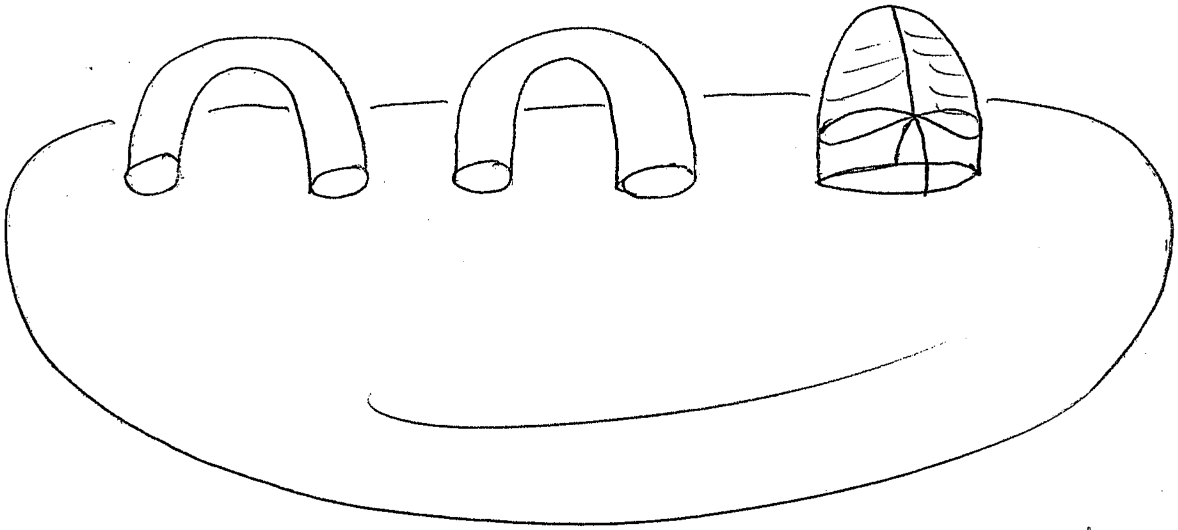
Since $\mathbb{P} \cong$ Mobius $\cup$ Disk, taking connected sum with $\mathbb{P}$ is same as cutting a single hole in the surface & then gluing a Mobius strip to the boundary of the hole.



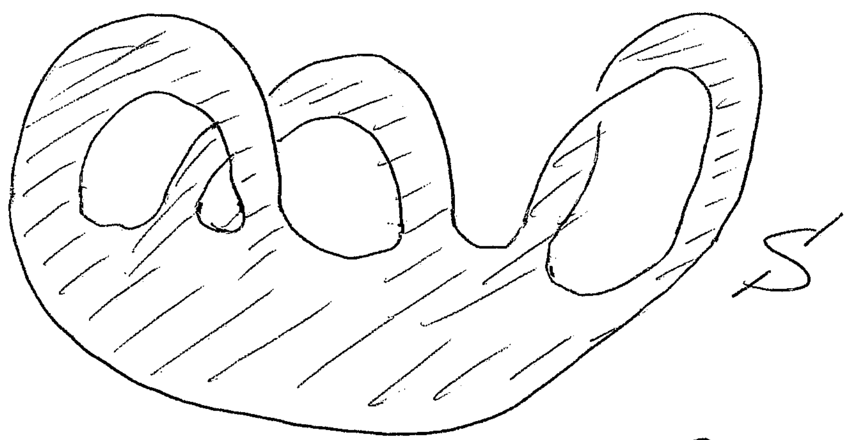
This operation is called "adding a cross-cap".
Sometimes you will see a singular picture of the Mobius strip:



This singular Mobius with circular boundary is also called a cross-cap.



$S_{2,1}$ = sphere with two handles and one cross-cap.

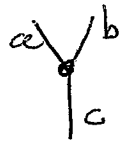


__Exercise__: $S = S_{n,m}$

Find $n =$ ____.

$m =$ ____.

Penrose Formula



$= \varepsilon_{abc}$ where "epsilon tensor"

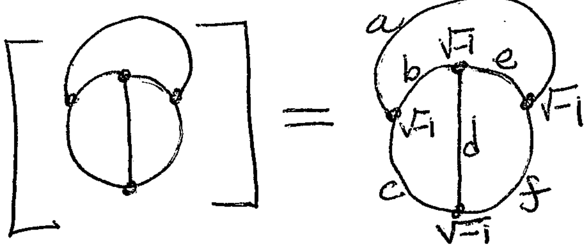
$$\begin{cases} \varepsilon_{123} = 1, & \varepsilon_{213} = -1 \\ \varepsilon_{312} = 1, & \varepsilon_{321} = -1 \\ \varepsilon_{231} = 1, & \varepsilon_{132} = -1 \end{cases}$$

Γ a trivalent plane graph.

Define $[\Gamma] = \sum_{a,b,c,\ldots \in \{1,2,3\}} \prod (\sqrt{-1})\varepsilon_{abc}$ where

we take one epsilon tensor for each vertex of Γ, and label all the edges of the graph differently.

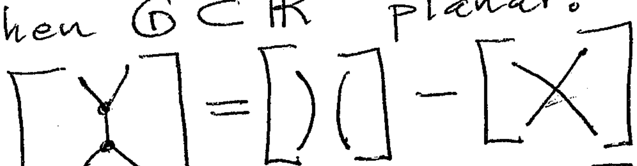
e.g.



$$= \sum_{a,b,c,d,e,f} (\sqrt{-1}\,\varepsilon_{abc})(\sqrt{-1}\,\varepsilon_{bed})(\sqrt{-1}\,\varepsilon_{eef})(\sqrt{-1}\,\varepsilon_{cdf}).$$

__Thm.__ (a) $[\Gamma]$ = # of edge 3-colorings of Γ (3 distinct colors per vertex) when $\Gamma \subset \mathbb{R}^2$ planar.

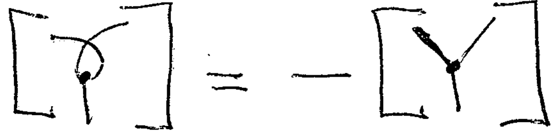
(b)



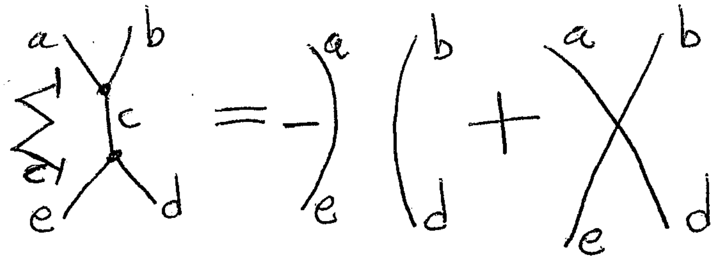
where $[O\,\Gamma] = 3[\Gamma]$

$[O] = 3.$

(c)



(b) comes from the "epsilon identity"

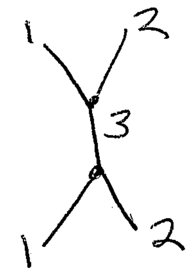


(sign switch because not using $\sqrt{-1}$ here.)

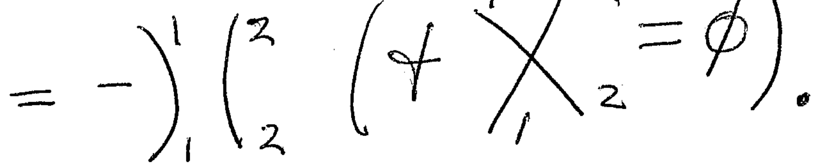
$$\sum_c \varepsilon_{abc}\varepsilon_{cde} = -\delta^a_e\,\delta^b_d + \delta^a_d\,\delta^b_e$$

$$\delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}.$$

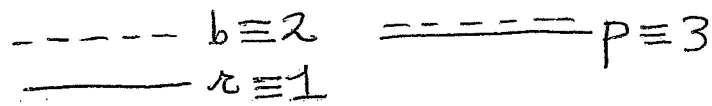
e.g.



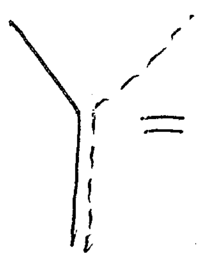
$= \varepsilon_{123}\varepsilon_{132} = (+1)(-1)$



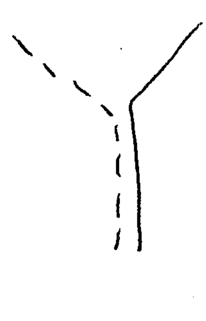
(a) comes from



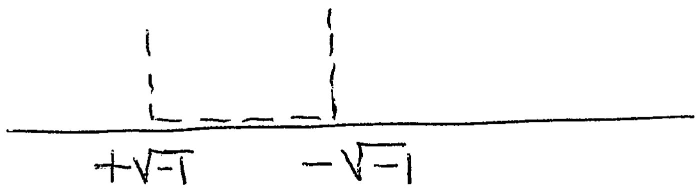
so



$= \varepsilon_{rpb} = \varepsilon_{123} = +1$

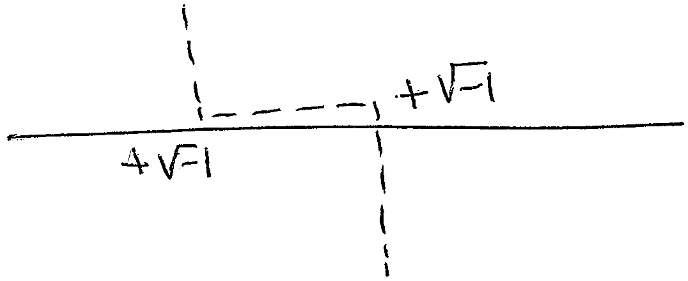


$= \varepsilon_{brp} = \varepsilon_{213} = -1$

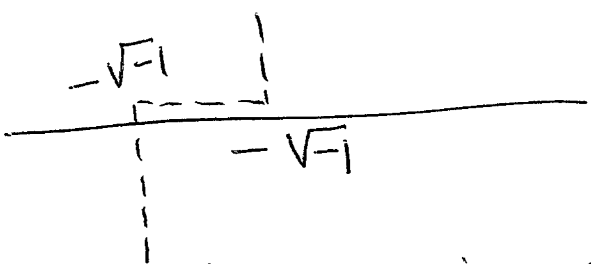


$(+\sqrt{-1})(-\sqrt{-1}) = +1$

each bounce contributes $+1$



$(+\sqrt{-1})(+\sqrt{-1}) = -1.$

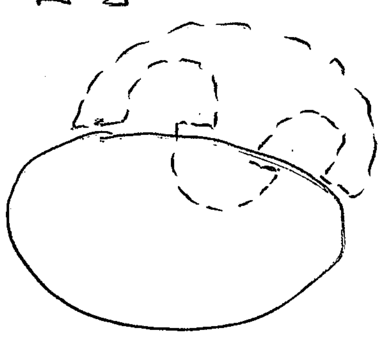


$(-\sqrt{-1})(-\sqrt{-1}) = -1.$

each crossing contributes $-1.$

$\therefore$ each $\prod(\sqrt{-1}\,\varepsilon_{abc})$ for a given coloring contributes $(-1)^{\#\text{crossings}}$.
But # crossings is even (by Jordan curve theorem) & $\therefore$ each product contributes $+1$ & so [G] counts colorings.

e.g.



contributes $(-1)^2 = +1.$ //

(c) is implied by $\varepsilon_{abc} = -\varepsilon_{bac}$. //