More Matrix Algebra (Mostly 2×2) + Graphs

$\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, <u>2×2 determinant</u>

<u>Fact</u>: If $\Delta = \mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$, then

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, and

$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$.

<u>Exercise</u>: Check that $AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

In fact, an $n \times n$ matrix A is invertible $\Leftrightarrow \mathrm{Det}(A) \neq 0$.
(The determinant function generalizes to $n \times n$ matrices, as we'll see below.)

<u>Exercise</u>. Let A and B be 2×2 matrices. Verify that $\mathrm{Det}(AB) = \mathrm{Det}(A)\mathrm{Det}(B)$.
(This also generalizes to $n \times n$ matrices.)

<u>Definition</u>. A vector $\vec{v}$ is said to be an <u>eigenvector</u> for an $n \times n$ matrix A if $A\vec{v} = \lambda\vec{v}$ for some number λ and $\vec{v} \neq 0$. λ is said to be an <u>eigenvalue of A</u> if this happens.

[For example, $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. Then $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus 3 and 4 are eigenvalues of A and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors.]

The <u>spectrum of A</u> is its set of eigenvalues.

<u>Theoretical Example</u>. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the eigenvalues of A. Want $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\lambda \in \mathbb{R}$ (or possibly $\mathbb{C}$; $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers)

so that

$$A\vec{v} = \lambda\vec{v}$$

$$\Leftrightarrow \quad A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$\Leftrightarrow \quad (A - \lambda I)\vec{v} = \vec{0}.$$

<u>Now note</u>: If $(A-\lambda I) = B$ is invertible, then we can multiply both sides of this equation by B^{-1}:

$$B\vec{v} = \vec{0}$$

$$B^{-1}B\vec{v} = B^{-1}\vec{0} = \vec{0}$$

$$\Rightarrow \quad \vec{v} = \vec{0}.$$

Since we want solutions where $\vec{v} \neq 0$, we conclude that <u>we</u> <u>need</u>

$(A - \lambda I)$ <u>not</u> invertible!

$$\Updownarrow$$

$$\boxed{\mathrm{Det}(A - \lambda I) = 0}$$

$$\mathrm{Det}(A - \lambda I) = \mathrm{Det}\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$\boxed{C(A,\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)}$$

The roots of the polynomial $C(A,\lambda)$ [<u>the characteristic polynomial of A</u>] are the possible eigenvalues of A.

<u>Exercise</u>. (a) Find the eigenvalues & char poly for $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ and for $A' = \begin{pmatrix} [illegible] & [illegible] \\ [illegible] & 0 \end{pmatrix}$.

(b) Prove that if P is invertible 2×2 matrix, A any 2×2 matrix, then $C(P^{-1}AP, \lambda) = C(A, \lambda)$.

Partial Solution. (a) $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$

$$C_A = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3).$$

Thus A has spectrum $\{2, 3\}$.

(b) $$\mathrm{Det}(P^{-1}AP - \lambda I) = \mathrm{Det}(P^{-1}AP - \lambda P^{-1}P)$$
$$= \mathrm{Det}(P^{-1}(AP - \lambda P))$$
$$= \mathrm{Det}(P^{-1}(A - \lambda I)P)$$
$$= \mathrm{Det}(P^{-1})\,\mathrm{Det}(A-\lambda I)\,\mathrm{Det}(P)$$
$$= \mathrm{Det}(P^{-1})\,\mathrm{Det}(P)\,\mathrm{Det}(A - \lambda I)$$
$$= \mathrm{Det}(I)\,\mathrm{Det}(A - \lambda I)$$
$$= \mathrm{Det}(A - \lambda I) \quad // \quad (\mathrm{Det}(I) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1)$$

Go back to part (a) and find eigenvectors for A. (i) $A\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (A - 2I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x + 2y = 0$$

e.g. $y = 1$, $x = -2$ $\quad \boxed{\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \vec{v}_1}$

$$A\vec{v}_1 = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} \checkmark$$

(ii) $A\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow (A - 3I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x + y = 0$$

$\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}.$ $\quad A\vec{v}_2 = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} \checkmark$

Now observe: $P = (\vec{v}_1 \ \vec{v}_2)$ columns.

$$\Rightarrow \quad AP = (\lambda_1\vec{v}_1 \ \ \lambda_2\vec{v}_2) \qquad \lambda_1 = 2, \ \lambda_2 = 3$$

$$\Rightarrow \quad AP = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

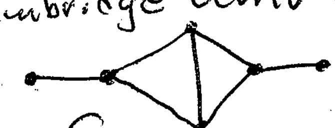
You will find that if a 2×2 matrix A has distinct eigenvalues λ_1, λ_2
Then corresponding eigenvectors $\vec{v}_1, \vec{v}_2$ s.t. $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ are linearly independent (in the sense that one is not a multiple of the other) and if you form $P = (\vec{v}_1\ \vec{v}_2)$, the matrix with columns $\vec{v}_1$ and $\vec{v}_2$, then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

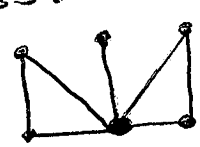
We say that A is diagonalizable.

Definition. The spectrum of a graph Γ is the spectrum of its adjacency matrix $A(\Gamma)$.

There exist non-isomorphic graphs with the same spectrum. But the problem of finding all graphs with a given spectrum is unsolved.
The following example is from "Algebraic Graph Theory" by N. Biggs Cambridge Univ Press.

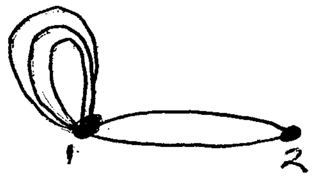


Γ



Γ'

Γ and Γ' have the same characteristic polynomial:
$$\lambda^6 - 7\lambda^4 - 4\lambda^3 + 7\lambda^2 + 4\lambda - 1$$
and hence the same spectrum.
Certainly Γ and Γ' are not isomorphic.

Example. 

G

$$A(G) = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$$

Since the powers of the adjacency matrix catalog the walks on G, we would like to find a formula for A^n.
If we can find P invertible and eigenvalues λ, μ s.t. $P^{-1}AP = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

Then $(P^{-1}AP)^n = P^{-1}A^nP = \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix}$.

So $\underline{A^n = P\begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix}P^{-1}}$.

$c_A(\lambda) = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4)$.

Let $\lambda = -1$, $\mu = 4$.

$$A - \lambda I = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} : \vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$A - \mu I = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} : \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathrm{Det}(P) = -1-4 = -5$$

$$P^{-1} = \frac{1}{(-5)}\begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}.$$

$$A^n = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 4^n \end{pmatrix}\frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} (-1)^{n+1} & 2\times 4^n \\ 2\times(-1)^n & 4^n \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$A^n = \frac{1}{5}\begin{pmatrix} (-1)^n + 4^{n+1} & 2\times(-1)^{n+1} + 2\times 4^n \\ 2\times(-1)^{n+1} + 2\times 4^n & 4\times(-1)^n + 4^n \end{pmatrix}$$

Here is another notion.

Let $W^{(n)}_{ij}$ = the number of walks of length n in G from node i to node j.

Define a formal power series

$$\left(\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}\right) \quad \Gamma_{ij}(t) = \delta_{ij} + W^{(1)}_{ij} t + W^{(2)}_{ij} t^2 + W^{(3)}_{ij} t^3 + \cdots$$

then since $W^{(n)}_{ij} = (A^n)_{ij}$ we have

$$\Gamma_{ij}(t) = A^0_{ij} + A^1_{ij} t + A^2_{ij} t^2 + \cdots$$

$$= (I + At + A^2t^2 + A^3t^3 + \cdots)_{ij}$$

Thus we should look at the formal matrix series

$$\Gamma(t) = I + At + A^2t^2 + A^3t^3 + \cdots$$

$$\Gamma(t) = \frac{I}{I - At} = (I - At)^{-1}.$$

$$(I - At) = -t(-t^{-1}I + A) = -t\left(A - \left(\tfrac{1}{t}\right)I\right)$$

Thus $I - At$ is invertible for formal t or for $(1/t)$ not an eigenvalue of A).

Apply to our example:

$$I - At = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3t & 2t \\ 2t & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-3t & -2t \\ -2t & 1 \end{pmatrix}$$

$$Det = |I - At| = \begin{vmatrix} 1-3t & -2t \\ -2t & 1 \end{vmatrix} = 1-3t-4t^2$$

$$\Gamma(t) = (I - At)^{-1} = \frac{1}{1-3t-4t^2}\begin{pmatrix} 1 & 2t \\ 2t & 1-3t \end{pmatrix}$$

Generating Function for walks from node 1 to node 1 in 

$$= \begin{pmatrix} \dfrac{1}{1-3t-4t^2} & \dfrac{2t}{1-3t-4t^2} \\[2ex] \dfrac{2t}{1-3t-4t^2} & \dfrac{1-3t}{1-3t-4t^2} \end{pmatrix}$$

So this means that

$$\frac{1}{1-3t-4t^2} = 1 + \sum_{n=1}^{\infty}\left[\frac{(-1)^n + 4^{n+1}}{5}\right] t^n$$

$$\frac{1}{1-3t+4t^2} = 1 + 3t + 13t^2 + 51t^3 + \cdots$$

Try this out! For example look at the long division:

$$\begin{array}{r|l} & 1+3t+13t^2+\cdots \\ \hline 1-3t-4t & 1 \\ & \underline{1-3t-4t^2} \\ & 3t+4t^2 \\ & \underline{3t-9t^2+12t^3} \\ & 13t^2-12t^3 \\ & \underline{13t^2-39t^3-52t^4} \\ & 51t^3+52t^4 \end{array}$$

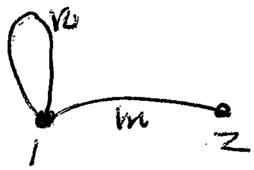
$1-3t-4t^2$

Exercises.

1.  $A(G) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$

G

Apply the methods in these notes to this graph.
Find characteristic polynomial, spectrum, eigenvalues, formulas for walks, generating functions.

2.  $A(G(n,m)) = \begin{pmatrix} n & m \\ m & 0 \end{pmatrix}.$

$G(n,m)$

Do as much as you can with this class of graphs. (n loops at 1, m edges from 1 to 2.).

3. Choose your own graph and analyze it in similar fashion.