Cemela Summer School Mathematics as language Fact or Metaphor?

John T. Baldwin

June 25, 2007

Goals

Formal languages arose to remedy the lack of precision in natural language.

1. Motivate with classroom examples the reasons for developing a formal language for mathematics.
2. Interweave the definition of a first order language adequate for mathematics
3. The interplay between natural language, 'regimented language', and formal language

Outline

Let M be Mary!

Problem

I went to Pompeii and bought the same number of salads and small pizzas. Salads cost two dollars each and pizzas cost six dollars each. I spent \$40 all together. Assume that the equation $2S + 6P = 40$ is correct.

What is wrong with the following reasoning? Be as detailed as possible. How would you try to help a student who made this mistake?

Solution

Then,

$$2S + 6P = 40.$$

Since $S = P$, I can write

$$2P + 6P = 40.$$

So

$$8P = 40.$$

The last equation says 8 pizzas is equal to \$40 so each pizza costs \$5.

Reprise

1. structures and languages;
2. the compositional theory of truth;
3. defined the truth of a sentence in a structure.
4. discussed the properties of equality and equality axioms.

Truth and Validity

We have defined $M \models \phi$.
But what does it mean to say ϕ is true?!

Give an example of a sentence ϕ and models M_1 and M_2 such that $M_1 \models \phi$ and $M_2 \models \neg\phi$.

Truth and Validity

We have defined $M \models \phi$.
But what does it mean to say ϕ is true?!

Give an example of a sentence ϕ and models M_1 and M_2 such that $M_1 \models \phi$ and $M_2 \models \neg\phi$.

Validity

The sentence ϕ is valid if it is true in every structure.

Why proof

Why do we give proofs?

Why proof

Why do we give proofs?

1. to understand why!
2. to organize knowledge and make it easier to remember

Why proof

Why do we give proofs?

1. to understand why!
2. to organize knowledge and make it easier to remember
3. to obtain certainty

A proof system

Logical Axioms

For any formula ϕ: $\phi \vee \neg\phi$.
The equality axioms.
$\phi_x(a) \rightarrow (\exists x)\phi$.

Inference rules

- Expansion:Infer $\phi \vee \psi$ from ψ
- Contraction:Infer ψ from $\psi \vee \psi$
- Associative: Infer $(\phi \vee \psi) \vee \chi$ from $\phi \vee (\psi \vee \chi)$
- Cut: Infer $\phi \vee \psi$ from $\phi \vee \chi$ and $\chi \rightarrow \psi$
- Exists introduction: If x is not free in ϕ, infer $(\exists x)\psi \rightarrow \phi$ from $\psi \rightarrow \phi$.

formal proof

A formal proof from a set of axioms Φ
is a sequence of wff's such that each one

1. is a member of Φ
2. or is a logical axiom
3. or follows from earlier lines by a rule of inference

The completeness theorem

Gödel I

There is a proof of ψ if and only ψ is valid.

There is a proof of ψ from Φ if and only ψ is true in every structure that satisfies each member of Φ.

The incompleteness theorem

Gödel II

There is no effective way to decide whether a sentence ϕ is valid.

The inerrancy of mathematics

There is a procedure to check a proof is correct.
There is no procedure to check if a sentence is valid.
But the valid sentences are not interesting anyhow.
To actually encode mathematics, add nonlogical axioms:

Some important sets of axioms

1. axioms for arithmetic
2. Axioms for the real field $(\Re, +, \times, <, = 0, 1)$
3. axioms for set theory

Thus the 'inerrant' part of mathematics becomes the logical deductions. It is essential to make your hypotheses and conclusions explicit.

The Moses Analysis: General

1. people talk (natural language)
2. feature talk (regimented language)
3. math talk (formal language)

A Socratic Dialogue

People Talk

Which costs less, a pizza or a salad?

People Talk

Which costs less, a pizza or a salad?

A salad costs less than a pizza

Features

What feature of pizza and salad are we talking about?

Features

What feature of pizza and salad are we talking about?

Their cost

Feature Talk

Write a sentence describing this situation beginning

The cost of a salad

Feature Talk

Write a sentence describing this situation beginning

The cost of a salad

is less than the cost of a pizza.

People Talk vrs Feature Talk

Where is the information about price encoded in these two statements?

1. A salad costs less than a pizza.
2. The cost of a salad is less than the cost of a pizza.

People Talk vrs Feature Talk

Where is the information about price encoded in these two statements?

1. A salad costs less than a pizza.
2. The cost of a salad is less than the cost of a pizza.

It moves from verb to noun.

People Talk vrs Feature Talk

Where is the information about price encoded in these two statements?

1. A salad costs less than a pizza.
2. The cost of a salad is less than the cost of a pizza.

It moves from verb to noun.
Let's abbreviate 2):

C(S) is less than C(P).

How much?

How much less does a salad cost?

How much?

How much less does a salad cost?

$4 less

How much?

How much less does a salad cost?

$4 less

people talk: A salad costs $4 less than a pizza.

feature talk: C(S) is $4 less than C(P).

Syntactical Observation

English adds the amount of change to the linking phrase.

Feature talk puts it in a name position.
Thus in feature talk the same verb is used in different situations; it becomes a relation (eventually $<, >$ or $=$ and the noun becomes a variable.

Mathematical Language

C(S) compared to C(P) is \$4 less.

Abbreviate:

C(S) c/t C(P) is \$4 less.

3 variants

C(S) c/t C(P) is \$4 less.

C(S) c/t C(P) is $^{-}$\$4.

$C(S) - C(P) =^{-}$ \$4.

Vectors for K-8 ???

Positive numbers wed direction and quantity.

They are displacements

Vectors for K-8 ???

Positive numbers wed direction and quantity.

They are displacements

Moses continues the analysis with trips on the MTA.

Disposable materials for students are located at http://www.algebra.org/ (programs/curriculum development/algebra)

Context

As in the logical approach I outlined at the beginning:

vocabulary: relations $<, =$, functions $+, -$

structure is $(Z, +, -)$

Two metaphors

The primary school metaphor for subtraction is ‘take away’.

The algebra metaphor for subtraction is ‘compared to’.

Reflection

One should not justify 'borrowing' by:

Since 8 is greater than 7 we cannot subtract (take away) 8 from 7.

Reflection

One should not justify 'borrowing' by:

Since 8 is greater than 7 we cannot subtract (take away) 8 from 7.

Note the statement is true in 2nd grade because the term $8 - 7$ cannot be interpreted in $(N, +, 1, -)$.

Many algebra books have silly statement about 'closure' under the operations. The authors didn't realize that the closure condition was redundant when the intended interpretation is the real or rational numbers.

Aside

Moses's example is about the height of the boys: Coast to Coast (CTC) and Watch Me.

What are three ways this example is different from mine?

parsing a non-native tongue

A rhombus is four sided figure with all sides of the same length

Which of 1-4 is not true in every rhombus?

1. The two diagonals have the same length.
2. Each diagonal bisects the two angles of the rhombus
3. The two diagonals are perpendicular.
4. The opposite angles have the same measure
5. All of 1-4 are true in every rhombus.

parsing a non-native tongue

A rhombus is four sided figure with all sides of the same length

Which of 1-4 is not true in every rhombus?

1. The two diagonals have the same length.
2. Each diagonal bisects the two angles of the rhombus
3. The two diagonals are perpendicular.
4. The opposite angles have the same measure
5. All of 1-4 are true in every rhombus.

Which of 1-4 is not true (in every rhombus)?

Let's take a poll?

Dr. Math,
What is the correct definition for a trapezoid? And why?
My questions come from the Math Department at Carroll Middle School in SouthLake, Texas. Two of the math teachers have found well-known publications with very different definitions for a Trapezoid.
1) Trapezoid: Quadrilateral with at least 1 pair of sides parallel.
2) Trapezoid: A trapezoid is a quadrilateral with exactly one pair of parallel sides.

Trape whatzis

What is a trapezoid, a trapezium?

Trape whatzis

What is a trapezoid, a trapezium?

A trapezoid (in North America) or trapezium (in Britain and elsewhere) is a quadrilateral, which is defined as a shape with four sides, which has a pair of parallel sides. Some authors define it as a quadrilateral having exactly one pair of parallel sides, so as to exclude parallelograms.
The exactly opposite concept, a quadrilateral that has no parallel sides, is referred to as a trapezium in North America, and as a trapezoid in Britain and elsewhere. (Wikipedia ?!?!?!)

defining Definition

What is a definition?

defining Definition

What is a definition?

An abbreviation

Definitions

Can a definition be right or wrong?

Define: prime number.

Definitions

Can a definition be right or wrong?

Define: prime number. Is 1 prime?
Is it just a matter of convention?

Theorem

Every natural number can be uniquely written as a product of prime numbers.

$$n = p_1^{k_1} \dots p_m^{k_m}$$

Thesis

Formal language provides a precise way to describe mathematical objects.
This ideal is a powerful tool for analyzing curriculum and discourse.