M IS NOT MARY: VARIABLES FROM GRADE 3 TO 13

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In this article we consider the development of the notion of variable and precursors of that notion by the examination of several problems. These problems are studied in specific contexts. The 'pizza problem' was adapted by Baldwin from an example in [LM99] and was used in a secondary math methods course. The 'chicken problem' is presented as a cartoon and a group of students were asked to find the weight of three chickens knowing the weight of each pair. We examine both the role of language and age in interpreting the problem and compare possible solution methods. The 'fence problem' is an old standard. We contrast two correct solutions with one abortive attempt by a student in a first university course for future elementary teachers. Finally, the sink problem is a standard sort of mixture problem. We discuss the approach taken to it in a first year of university intermediate algebra course taught-jointly with an introduction to chemistry. In each of these examples we will stress the importance of a precise specification of what the variable represents. We will clarify further the goals of comparing these problems after explaining the connections between algebraic expressions and numbers in the next section.

Like most problems in school mathematics, these do not arise in nature. They have been contrived for pedagogical purposes. We discuss these purposes in Section 4.

1. WHAT IS A VARIABLE?

The term 'variable' is used in many ways. The words independent and dependent variable are introduced to describe the argument and range of a function. This notion of variable developed since the 18th century in an attempt to explicate calculus. In many different contexts a variable is a symbol that can be replaced by a number. This usage, which we refer to as the *substitutional approach* is much earlier; stemming from at least the 14th century. It encompasses the first and is actually necessary to describe the computation of an arithmetical function.

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As noted, the general notion 'variable' has a long and complex history. In this note, variable refers (as in most mathematics) to a symbol such as x, y, $\ldots$. There are three components to the use of such a symbol: an abstract expression or equation containing the symbol, a range of numbers for the variable to represent, and a verbal assignment of the meaning of the variable (Mary's age in years). The last component is intrinsically tied to assignment of units.

We describe below, specifically for the algebra of the real numbers, how to interpret various uses of variable. This description is of course simply one instance of a general procedure that can be found in any undergraduate text in symbolic logic. And rather than restricting to number we can consider other structures such as geometries, graphs etc. This analysis is the result of investigations by such philosophers and mathematicians as Pierce, Frege, Hilbert, Löwenheim, Skolem, Gödel, and Tarski. There is no thought that a fully formal explanation of the meaning of variables as begun in this section is part of the K-12 curriculum; rather it is a way to describe one aspect of that curriculum. Note however that the description we give below of the interpretation of expressions and equations is implicit in many high school algebra books, e.g. chapter 2 of [Cor09].

In the simplest sense a function is a rule that assigns to each member of its domain a unique value. Thus the domain might be the words (strings of letters) in English and the function f could assign to each word the number of distinct letters occurring in it. Frequently, we write $f(x)$ rather than f although the x adds no information. Karl Menger [Men53] argued powerfully but futilely against writing the x more than 50 years ago [1]. The development of intuition for the notion of function is an important subject for study but not one we address here. Rather we are more concerned with the transition to writing expressions for functions.

'Algebra' generally refers to contexts where the functions takes a set of numbers (e.g. the reals), which is equipped with operations $(+, \cdot)$, to itself. One can describe these functions without variables. We might write A^3 for the 'add three' function. This kind of idea has been explored extensively for developing function intuition in children (e.g. []PAGE- maneuvers on lattices, []ALGEBRA PROJECT. But when the function is defined by a more complicated combination of the operations on the domain, it is useful to introduce a symbol such as x to represent the argument of the function. We illustrate the versatility of this notion in the following examples.

In arithmetic, expressions are formal strings of symbols that are either names for numbers, or names for the fundamental arithmetical operations such as addition or multiplication. We explain below how to assign meaning to such

[1] Menger distinguishes 'scientific' and 'pure' concepts of variable; our discussion of verbal description and units corresponds to his 'scientific'

expressions. In arithmetic we write expressions such as $1+1$ [2] and equations such as

$$3+4=5+2. \tag{1}$$

We have a set of numbers, say the real numbers, in mind and the symbols $1, 2, 3, \ldots, 1/2, 1/3, \pi \ldots$ naturally denote particular real numbers. And an equation is either true (Equation 1) or false:

$$3+4=5+3.$$

In studying algebra, we introduce a new group of symbols, called variables; they usually are letters such as $x, y, z, \ldots$.

This allows us to write new expressions [3] such as $x+3$ or $3x^2+5x+2$ and new equations such as

(1) $y = x+3$
(2) $3x^2+5x+2=0$
(3) $x^2+y^2=1$
(4) $b = kd$.

There is no quantification involved in these expressions and equations; we say the variables are *free*. These equations appear similar but are used in different ways. We will discuss the four uses in turn. In each of these equations, the variable are *free* (not quantified). Equations with free variable determine relations on the real numbers.

(1) Life is now more complicated than when we considered arithmetic. The expression $x+3$ does not denote a number; for each particular value that is substituted for x, we get another number (the first plus 3). An expression like $x+3$ determines (or represents) a function. In fact, we take advantage of this and write the equation $y = x+3$. This equation is neither true nor false. Rather, it defines a subset of $\Re\times\Re$: the collection of pairs $\langle a, b\rangle$ such that $b = a+3$. And so we compute the 'add 3' function by substituting a value for x and evaluating the expression.
(2) The *solution set* of an equation in one variable is a set of real numbers. That is, $3x^2+5x+2=0$ defines the subset of those numbers a such that $3a^2+5a+2=0$. Now since the real numbers satisfy the distributive law: $3a^2+5a+2=(3a+2)(a+1)$. And since the real numbers have no non-trivial zero divisors $(3a+2)(a+1)=0$ implies that $3a+2=0$

[2]Technically, the 1 in Equation 1 is a numeral, a name for a number. Trying to make this distinction in the lower grades was one of the notorious follies of the 'new math'. But it is essential in algebra to distinguish between expressions or equations on the one hand and numbers on the other.

[3]In fact if introduce x^n as an abbreviation for the product of n x', we have defined the class of polynomials as done in high school algebra.

or $a + 1 = 0$. So the only two numbers that satisfy the given equation are $-2/3$ and -1. So $3x^2 + 5x + 2 = 0$ is a fancy way to describe the set $\{-2/3, -1\}$. In this context, the word *unknown* is often used instead of variable. We are trying to find what values can be substituted for x to make the equation true.

(3) The equation $x^2 + y^2 = 1$, is a less trivial example. It defines the unit circle; all pairs of numbers (a, b) such that $a^2 + b^2 = 1$.

How does the word 'vary' enter the picture? In the first context we *vary* the argument by choosing which number to substitute for x and then we compute the value of 'add 3' at that argument.

(4) Consider the bouncing ball experiment. A ball is dropped from a various heights and each time we measure the height to which it bounces back. We collect data and to analyze it we fix the following vocabulary. The 'manipulated variable', d, is drop height - the distance above the ground from which we drop the ball. The 'responding variable', b, is bounce height - the distance above the ground the ball rises to. (I have data to fill in here; this is a standard TIMS experiment.) Suppose that the data shows the bounce height is 3/4 of the drop height. How do we represent that information as an equation? We write

$$b = \frac{3}{4}d$$

and interpret this equation exactly as in case 1). But this example illustrates the flexibility of our notation. $b = \frac{3}{4}d$, is the result of substituting $3/4$ for the variable k in the equation in three variables $b = kd$. For any particular ball, we find that the 'coefficient of resiliency' k is constant. Thus we have a family of equations with the *parameter* k; we say the bounce height is *proportional* to the drop height.

So our analysis of the bouncing ball represents a more general phenomena. We have an equation in several variables (for simplicity: k, d, b); thus it defines a subset of $\Re^3$. For any particular choice (substitution) of a value for k, we get an equation with a 'manipulated' (or independent) variable d and a 'responding' (or dependent) variable b. To describe the graphs of these equations we consider substitutions of real numbers into the equation $b = kd$; these give us a subset of $\Re^2$.

A system of equations defines the intersection of the sets defined by the individual equations.

The equation $x(y+z) = xy+xz$ is a problematic notation. If we interpret it in the same way as the examples above we see that it defines $\Re \times \Re \times \Re$. So we really meant to write:

$$(\forall x)(\forall y)(\forall z)x(y + z) = xy + xz.$$

This closed (no free variables) sentence is true because it is true no matter what triple of real numbers is substituted for the variable. We say that the universal quantifiers $\forall$ have bound the variables. Quantification plays an important role as soon as such notions as limit are defined. But for algebra only (systems of) equations with no bound variable or universal quantifications of equations (laws or axioms) are important. And often the universal quantifiers are omitted for convenience despite the ambiguity. Our discussion here is the beginning of the definition by induction of truth for arbitrary sentences in first order logic (thus involving nesting $\forall$ and $\exists$). The further development does not appear in 'algebra' and so is omitted here.

The first three problems we discuss fall into class 2); the third unites class 2) and class 4). The last three problems can be represented as systems of linear equations. Indeed the chicken and fence problems are represented in matrix form by

$$AX = C$$

where the 3×3 matrix A is the same in both problems, although the constants C are different.

In all but the first problem, we examine arithmetic solution methods which do not require the use of a symbolic variable. We want to contrast the mental models involved in these concrete solution methods with the ones involved in algebraic solutions. Since the problems themselves are of little inherent use, we consider that the purpose for studying them in school is to develop intuitions that support solving a large class of problems. So our question is, 'Do the ways of thinking involved in the supposedly intuitive solutions help develop an understanding of variable that transfers to other situations?' This question underlies our study of each of the examples.

The solution of word problems involves three components. Algebraic expressions, the interpretation of equations as subsets of the real numbers, and the verbal description of the variable. With the formal development of the first two in hand, we consider in several instances the role of the choice of the verbal description.

2. PIZZA PROBLEM

This problem adapted to a local situation an example in [LM99] and has been given both on exams and for class discussion to several classes of students in Methods of Teaching Mathematics. These were either graduate students or advanced undergraduate preparing for secondary teaching.

Problem 2.1. I went to the Pompeii restaurant and bought the same number of salads and small pizzas. Salads cost two dollars each and pizzas cost six dollars each. I spent \$40 all together. Assume that the equation $2S + 6P = 40$ is correct.

Then,

$$2S + 6P = 40.$$

Since $S = P$, I can write

$$2P + 6P = 40.$$

So

$$8P = 40.$$

The last equation says 8 pizzas is equal to \$40 so each pizza costs \$5.

What is wrong with the above reasoning? Be as detailed as possible. How would you try to help a student who made this mistake?

In each case only about one-half of the students identified the source of the difficulty: P is the *number of pizzas* I bought; not the cost of the pizza, and not 'pizzas'. This problem proved an effective way of drawing the students attention to the need to identify the variable verbally and determine what set of numbers they range over.

Note that after applying the distributive law P is number of 'meals' bought. One advantage of algebra is that we do *not* have an assigned meaning for each variable that remains the same throughout the computation. And indeed, the example shows that this is often impossible, even in very simple situations.

To elaborate at bit, in the given equation $2S + 6P = 40$, the coefficients 2 and 6 indicate unit prices of salads and pizza each. 2 and 6 are not 2 dollars and 6 dollars; they represent 2 dollars/salads and 6 dollars/pizza. So, the addition of them doesn't result in 8 dollars; it is 8 dollars/(salads and pizza), or meals as we wrote above. For arithmetical reasoning, it is important to know that the sum of the unit prices of different units requires a new unit for the sum. If we continue this fine analysis, in the equations $2P + 6P = 40$ and $8P = 40$, each P refers to a different unit: the number of salads, the number of pizzas, and the number of (salads and pizza) in order. Again, one power of algebra is that we do not have to worry about such matters. If our formal algebraic reasoning is correct and we have assigned the proper meaning to P at the beginning of the calculation, the value of P is that meaning, the number of pizzas.

This problem involves only one variable; the rest concern systems of equations. But we began with it to demonstrate the difficulty relative experts have with interpreting word problems and properly describing the variables involved. This difficulty may be exacerbated by the use of problems with 'intuitive solutions' that do not support a proper understanding of variable.

3. CHICKEN PROBLEM

CEMELA (Center for the Mathematics Education of Latinos) is a consortium of four universities. The group in Chicago runs after-school program at a grade school with primarily bilingual students in Chicago. The students are presented with activities and encouraged to work on them; their work is facilitated by an undergraduate. The Chicken problem is in the appendix. The discussion below concerns roughly three minutes of tape. This tape is of 3rd graders in the CEMELA after-school program who are trying to solve this problem.

There are 85 short statements in the tape. In the following interpretative summary we identify the general flow of the discussion by referring to statement numbers and quote in full salient comments.

The students begin by identifying different aspects of the problem. In the first 17 speeches they note that there are three sizes of chickens and that the numbers are the weights of certain pairs. At speech 17, the student Alf asks 'You need to add them all up?' In speeches 18 -24, the facilitator tries to elicit a strategy. Seizing on Alf announcing 24 in speech 25, the facilatator asks:

26: Cr: So if you add them all up, how many chickens do you have.

There is confusion, as Alf and then Alf and Ra reply 24.

There are two investigations taking place. 'How many chickens are there?' 'What do you get when you add em all up?"

The two questions are intermingled. And there are several answers proffered for each question.

In response to the question, 'how many chickens', one hears answers of three (the facilitator in lines 9, 10 and 24), six (Ra, (Alf, Ro in speeches 32,33) and, most tellingly, nine (Ra 36,37). The nine comes from the student Ra counting all the images on the page. The facilitator is seeking the answer, 'six' – the number of chicken images on the three cartoons that have the weights specified. And Ra agrees with this in line 45. Seeing that there are six images in these three pictures is the facilitator's strategy to get the students to realize that they must divide the sum of the three given weights by two. But it introduces the issue, 'what does the six represent'?

The numbers 24 , 34, 44 represent different students' opinions as to the total weight of chickens in the first three cartoons. Alf consistently says 24, beginning in speech 25. But while Ra agrees with him in line 29, by line 40 he changes to 34 and persists in this at least through line 52. He mocks Alf in line 48. (It appears that this over-count comes from adding the first row and the first column and then adding the those two sums together.)

What accounts for the different numbers? (It appears that Ra's over-count of 34 comes from adding the first row and the first column and then adding the those two sums together.) But in line 85, he suddenly says 44. Looking carefully at the tape and at the problem handed out suggests that this is misreading the initial exclamation point in the Spanish version a one in cartoons 2 and 3. The 34 perhaps come from the student adding both across the first row and down the first column.

It appears that these third graders don't understand the conventions for reading cartoons. The teacher might address that problem by having the students interpret the individual cartoons before trying to solve the problem. Or one might feel that the problem is simply 'too hard' for students of this age.

But the more important issue concerns the question "How many chickens are there?". It may be a reasonable strategy to elicit a solution to the problem.: Since six pictures of chickens appear in the cartoons whose weights total 24 kg, we have to divide 24 by 2. But it produces a confusing doubling of each chicken. We explore this issue further below.

The tape shows that when such problems are used in elementary school to develop students understanding of algebra, the teacher must have a very sensitive understanding of exactly what is happening in a solution.

It seems very important that these are *third* graders; the materials were taken from a 6th grade curriculum but have been used with all ages.

In particular, Gail Burril []Burril has reported that when used with secondary math methods class, the students set up a system of linear equations as below. But when she discussed the problem in lectures, mathematicians, gave the intuitive solution the undergraduate facilitator above was looking for. These are the appropriate equations:

$$A + B = 6 \tag{2}$$

$$B + C = 10 \tag{3}$$

$$A + C = 8 \tag{4}$$

where A, B, C are the weights of the small, medium, and large chicken, respectively. Even at this point, where we have reduced to equations, there are two alternatives. Students tend to solve these equations by blindly applying the method of Gaussian elimination, eliminating in pairs. But, recognizing the symmetry of the situation (each variable occurs twice), one might first see that the twice the sum of the three variable is 24 so the sum of the variable is 12 and then, e.g. since $A + B = 6$, C must be 6. This would represent insightful manipulation of the 'naked math' representation. It is crucial for this method that $2A$ is twice the weight of a small chicken, not two small chickens.

Problem 1. *Study the work of older children on the chicken problem. Does the issue of counting the chickens arise. Does this lead to confusion between 'chicken' and 'weight' as the variable?*

4. WORK PROBLEMS

Zal Usiskin [Usi80, Usi07] inveighs at length against traditional word problems in his articles on 'What should not be in the algebra curriculum ...!'

He writes, " The traditional word problems (coin, age, mixture, distance-rate-time, and digit) are in the curriculum because of a very valuable goal, the goal of translating from the real world into mathematics. But except for mixture problems, they do not help achieve that goal. In fact, they convince students that are no real applications of algebra because they are so ridiculous."

We largely agree with this critique. But, while 'translating from the real world into mathematics' is one purpose of these problems, there is a second: 'giving easily accessible examples of the use of variables'. That is, rather than demonstrations of the power of algebra, these problems can be seen of ways of making algebraic representations accessible. Of course, this goal is also defeated by unreasonable examples. But *pace* [**?**] BAGGETTEHRENFEUCHT on nickels, students have a sense of humor. And problems that are intentionally ridiculous have some redeeming social value. I am not sure whether that applies to the next problem. But it does illustrate several other ideas of this paper. It is perhaps crucial to note that this essay was written in the spring of 2008 when John McCain was clearly the Republican candidate for president but Hillary Clinton and Barack Obama were still contending for the Democratic nomination.

4.1. **The Fence problem.** This problem appeared in a course for future elementary school teachers from the text [Bec08]. We call the intended solution method, which is extensively used in the Singapore Curriculum, []SINGAPORE the 'strip method'. In each of these courses there is extended development of the strip method for problems of various sorts of which such work problems are among the most complicated. Abramovich and Nabors [AN97, AN97] elaborate the use of similar methods, which they dub enactive, using spreadsheet software.

Problem 4.1. Hillary and Barack can paint a fence in one hour.

So can Barack and John.

But Hillary and John take two hours.

How long does it take Hillary, Barack and John

So if we had two each of Hillary, Barack and John they would paint 2 1/2 fences in one hour.

Thus, the actual three can paint 5/4 of a fence in an hour.

And so it takes them 4/5 of an hour to paint the fence.

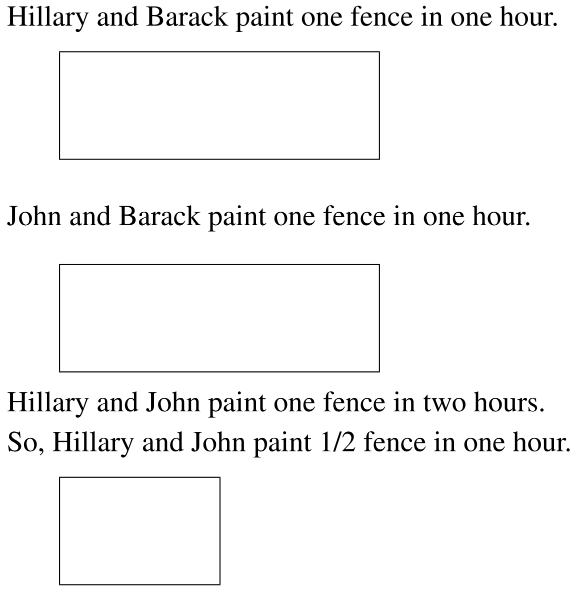

FIGURE 1. Strip Method

This is a concrete method of solution. Even after the use of the fraction strip method, there are two difficult steps in this solution. Notice again that, like the chicken problem it relies, on a doubling of the characters. The first is the assertion that if the 'doubled' actors can do the task in 2 1/2 hours it takes three people 5/4 of an hour. The second is the decision to invert the fraction in the last step. One can (Singapore does) teach arithmetic so this step is automatic. But it is a separate and difficult intuition.

The two papers [AN97, AN97] suggest how interesting questions in divisibility can be developed in this context: by asking when the solution is an integral number of days.

More important from our standpoint. There aren't really two Hilarys. Just as the chicken problem, the mental doubling is problematic.

We will discuss below the special conditions that make such a concrete solution possible. But first let's see what happens from a too-quick jump to forming equations.

Solution 4.2 (time equations)**.** A pre-service elementary teacher was given the problem in the 'a job of work form'. She attempted the problem by writing the following system of equations.

$$A + B = 1 \tag{5}$$
$$B + C = 1 \tag{6}$$
$$A + C = 2 \tag{7}$$

What is the difficulty? The variables seem to represent the amount of time taken by each person. But this is contradictory.

Solution 4.3 (From strips to equations)**.** The strips approach leads naturally to the equations:

$$A + B = 1 \tag{8}$$
$$B + C = 1 \tag{9}$$
$$A + C = 1/2 \tag{10}$$

What do the variables represent in this solution? The *amount of fence* each person does *in one hour*. The unit 'fences' is determined by inspecting the right hand side of the equation. We will consider these equations again after our next set of examples.

4.2. **Sink Problems.** The next series of problems are from a paired course in basic chemistry and intermediate algebra. The course was taught by a mathematician and a chemist. They attempt to put this kind of problem in a coherent framework through the use of functions. Note that Intermediate Algebra has essentially the content of Algebra II in the American high school. Nevertheless, it is the largest single course at most universities (even though frequently offered for no credit)REFERENCE?. The functions approach described below is not in general use in Intermediate Algebra, but was central in our course.

The problems and solutions described followed a lot of work on linear functions and then on linear equations. The function notation had been used in the class for some time before this problem was presented. When this problem was presented to a conference containing unilingual Spanish elementary teachers, several of them had trouble realizing that the proffered solution to the next problem was actually intended as a solution. This may have been just a language difficulty. But as they were talking with a translator, I think it more likely they foundered on the notation $H(t)$.

❙ Is this kind of reference to the conference permissible?

Problem 4.4 (A rate problem: Filling Sinks I)**.** The hot water tap delivers 3 quarts per minute; the cold water tap delivers 4 quarts per minute. If both taps are turned on how long does it take to fill a sink that holds 12 quarts?

Solution 4.5 (Functional solution)**.** In the this case we are given the two rates:

cold water: 3 quarts per minute

hot water: 4 quarts per minute

So in any t minutes, the cold water delivers $3t$ quarts:

$$C(t) = 3t$$

and the hot water delivers $4t$ quarts:

$$H(t) = 4t.$$

We are asked how long it takes for them together to fill one sink which holds 12 quarts. Let $T(t)$ be the amount of water delivered in t minutes.

Then

$$C(t) + H(t) = T(t).$$

And we are asked for what t, is $T(t) = 12$. That is,

$$C(t) + H(t) = 12$$

So, we must solve:

$$3t + 4t = 12.$$

Easily, $t = 12/7$ minutes.

The solution here combines the uses of variables in classes 4) and 2) of Section 1. First we have identified functions H, C, T that denote the amount of water delivered by the various taps after a given amount of time. Then, we have introduced a variable and formula to represent these functions symbolically. Finally, with this formalism, we use variable as in class 2) to complete the solution of the problem.

Sink problem I, with given rates was used as transition between rate problems in one variable and a method of solution of general work problems of the fence problem type. Here is an example of the general situation.

Problem 4.6 (A work problem: Filling Sinks II)**.** The hot water tap can fill the sink in 3 minutes; the cold water tap can fill it in 4 minutes. If both taps are turned on how long does it take to fill the sink.

Solution 4.7 (Work problem: rate solution)**.** To solve this problem, we need to be creative about rates. Instead of using 'natural' rates like 'quarts per minute', we invent a unit of sinks per minute.

Then we have the rates:

cold water: 1/4 sink per minute

hot water: 1/3 sink per minute

So in any t minutes, the hot water fills $t/3$ sinks and the cold water fills $t/4$ sinks. We are asked how long it takes for the two taps together to fill one sink.

$$C(t) + H(t) = 1$$

$$t/3 + t/4 = 1$$

$$t(1/3 + 1/4) = 1$$

$$\frac{7}{12}t = 1$$

$t = 12/7$ minutes.

This problem required the insight (recall of similar situations) to realize that the trick is to create an artificial rate of sinks per minute. With this insight it reduces to the previous case.

This raise a pedagogical issue. The last two problems were chosen with exactly the same parameters; would the point have been made better by using different numbers?

We return to the fence problem using the focus on rates from the sink problem.

Solution 4.8 (Fence problem: rate approach)**.** Let A, B and C be the rates in fences per hour at which Hillary, Barack, and John respectively paints.

Now

(11) $$A \cdot 1 + B \cdot 1 = 1$$

(12) $$B \cdot 1 + C \cdot 1 = 1$$

(13) $$A \cdot 2 + C \cdot 2 = 1$$

This of course yields an equivalent system to the fraction strip approach; but the method is uniform. In contrast to equations 8-10, the coefficients determined by the times given for the painting are explicitly represented.

The approach in Intermediate Algebra Course differs from the earlier problems and solutions in several important ways. The course was aimed a more advanced students and functional notation was introduced. One of Usiskin's complaints about word problems was addressed by using as actors the hot and cold water taps which can easily cooperate. This is an explicitly algebraic solution where the time variable is clearly identified. The effect of this formalization is seen most clearly in the fence problem. Instead of some 'copying of people, the amount of time that each person works is doubled. Similarly, this approach to the chicken problem makes explicit that it is the weight of the chicken that is doubled, not the chicken.

In the strip approach the variable was: the amount done *in* a unit time.

In the functions approach the variable was: the amount done *per* unit time.

The difference between these two notions is often blurred in speech– even by mathematicians. But the distinction is essential. The units of the variables (fences; fences per minute) are different.

By identifying the variable as a rate the fence problem is made part of a general pattern – rather than one more isolated technique.

5. SUMMARY

We have given three problems: chicken, fence, sink that can be solved as systems of linear equations. We have also presented other solution techniques.

We have described several methods of solving various problems. Some of these solutions are very particular to the problem in two senses. The context of the problem is very helpful in choosing the solution. The exact choice of numerical coefficients is essential to the solution method. That is, the fact that each variable occurs twice with coefficient 1 is crucial to the arithmetic solution technique for the chicken and fence problems. In solving the fence problem by rate approach, Solution 4.8, this computational trick is not available unless equation 13) is divided by 2.

Much effort has been fruitfully spent in the last few decades in stressing that many problems have different techniques of solution. It is equally important to realize that not all solutions are equal. Some are more efficient, more insightful, more general or more beautiful than others.

We argue that the problems we are considering are placed in the curriculum both to show that algebra connects to the real world and to prior arithmetic understandings. And to develop students understanding of variable and how to set up equations. We have questioned how various approaches contribute to the goals.

We certainly accept the notion that there may be 'developmentally appropriate' methods for the same problem. We raise the finer question. What is the 'developmentally appropriate' method at specific level for the class of problems considered here?

In several of the examples, we have been careful to indicate that when this material came up in the course, there had been substantial development of techniques related to the solution method suggested. The necessity for these developments present a key issue about the application of the analysis in this article. We have discussed connections between 'arithmetic' and 'algebraic' methods of solution. But this discussion is conducted with some worry that this issue is too fine. Given the varied students and situations, is the analysis suggested here a realistic element of curriculum design?

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