Ignorance and Innocence in the Teaching of Mathematics

Research and projects in school math education tend to begin with the assumption that the researcher knows all the mathematics he needs to know, and that his audience does, too. The research problem might be how to interest students in some (presumably well-known) body of mathematical knowledge, how to make them understand it, how to make them remember it and to use it. More commonly, in the United States, the research concentrates on how to strengthen a students deeper concepts of a mathematical truth, rather than some closely defined "body of knowledge", so that the student, thus equipped with "higher order thinking skills", will be able to discover for himself whatever procedures or connections a given real-life problem presents.

However all this might be, it is hard to imagine someone conducting a research project concerning mathematics teaching who would even consider the possibility that the mathematical lessons being conducted by his research subjects -- the actual material being taught, better or worse as the researcher is trying to determine -- are in fact not well understood by the teacher himself, let alone the researcher who is studying ways to enlighten or future teachers or writers of textbooks. Professors in the colleges that teach teachers are the presumed audience for such research, and certainly have experience in observing classroom teaching too.

Thus the author of a textbook instructing future teachers in how to go about teaching -- such books are sometimes called "methods" textbooks -- is unlikely to believe that he, himself, doesn't understand the mathematics involved. After all, this is material to be mastered by 8th grade children, or at most seniors in high school, and a professor in a teachers college, who is now writing books for the instruction of future teachers, has surely got beyond that level in the mere mathematics.

Members of the public, represented by editorialists in the newspapers, for example, read and write about the ever-current debates: whether children should or should not be required to memorize certain things, or whether classrooms should be more or less disciplined, or concentrated on practical or theoretical material. The editorialist some times goes so far as to discuss whether teachers are sufficiently credentialed, or experienced, but it would be rare indeed if someone should write about whether those doing the credentialing are themselves competent. How if the understanding of mathematics on the part of the judge of the educator is simply wrong? So wrong that what he is trying to put across in his "methods of teaching" textbook, or measuring in his researches on how children learn, simply cannot be understood because it is at bottom senseless?

Shall we credential teachers on how well they perform on examinations set and judged by professors who themselves are incompetent? As a corollary question, one might ask if there can be such a thing as competence in teaching, when the material putatively being transmitted is wrongly understood by the teacher.

That examples of this phenomenon were as much the rule as the exception in the period 1940-1950 will give some insight into the causes of the "The New Math" phenomenon that followed. In evaluating the way mathematics was being taught in the schools on the eve of The New Math, it will be worth while not only to examine what textbooks were saying, or school examinations examining, but also to see what future teachers were being taught by their own professors in the colleges. One can see from the texts of a few of these books what the picture of mathematics was in the minds of the educational elite of the time, and why, when mathematicians (as distinguished from professors of school mathematics education) began to take an interest in reform of school mathematics around 1950, their suggested remedies took the form they did. We will here present a few examples from the literature of the time.

Euclidean Geometry in 1900-1950

In a course in Euclidean geometry such as used to be given in the 10th grade, it was common in the year 1900 to present, or at least refer to, a "theory of limits" for use in proving such theorems as that a line parallel to one side of a triangle divides the other two sides proportionately. For teaching purposes in the schools an intelligent teacher need not be fully acquainted with all the properties of real numbers, or all of Euclid's Book V to provide a fair intuitive account of this theorem. Beginning with some easy lemmas about equidistant parallels cutting equal segments from transversals, one can convincingly show the theorem of proportionality for commensurable segments. That is, if in Figure 1

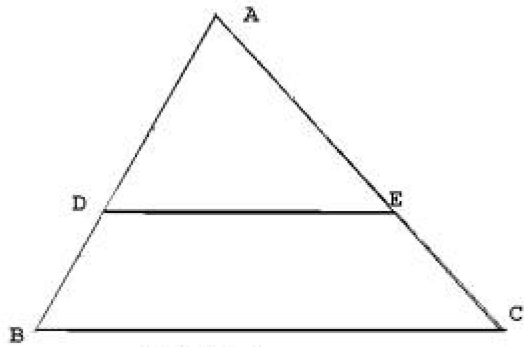

Figure 1

the line DE is parallel to BC, and if AD : DB is (say) 5 : 3, as indicated in Figure 2,

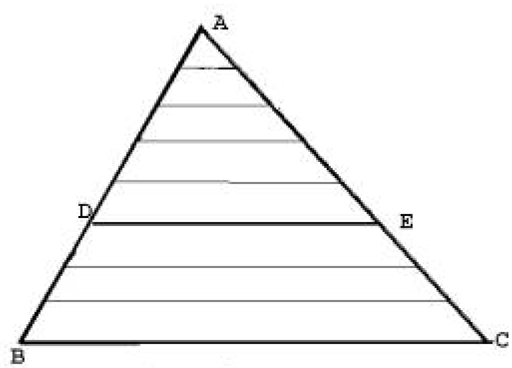

Figure 2

then AE : EC is easily shown also to be 5 : 3 by a standard Euclidean construction of parallel lines. The same is plainly true for any ratio m : n, if AD:DB happens to be m : n for some other whole numbers m and n, even very large ones. But since not every point D will divide the segment into two pieces of this nature, AD and DB being in such case called "incommensurable", the proof that AD : DB=AE : EC cannot use the facts about equally spaced parallels cutting off equal segments on all transversals, and in fact the very idea of a "ratio" AD : DB becomes problematic.

If ratios are construed in geometry as real numbers, rational in the commensurable case, the proof is completed by approximating a pair of possibly irrational quotients by rational ones representing parallels close to the one in question.

In Figure 1, then, if AD and DB have no common measure, we choose a tiny "atomic" part of AD such that there are (say) m such bits that do measure it, and then mark off as many of these little segments from D towards B as one can, say n of them, ending at a point B' from which the last possible such parallel to BC can be drawn, to intersect AB at B' and AC at C’. (Fig 3):

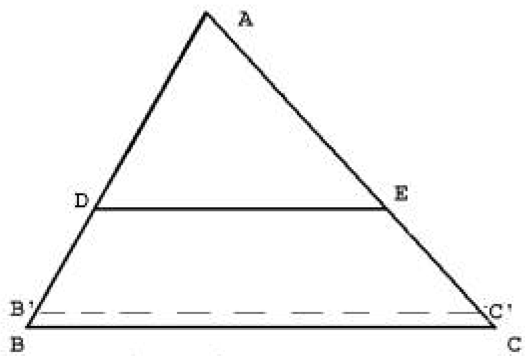

Figure 3

For the triangle AB'C' the theorem is true because one can merely count the equal segments: AD : DB' = AE : EC'. This being true for points B' and C' as close to B and C as desired, one says (grandly) that "in the limit" follows the truth of the proposition for the triangle ABC as well: DB' converges to DB and EC' converges to EC, so that ratios involving them converge also.

All this is far from rigorous, of course, and it replaces the Euclidean notion of ratio by our present-day notion of quotient of real numbers (which can only be fully understood via a development as difficult as the one employed by Euclid for its geometric counterpart); but the proof "by limits" can be enlightening at the high school level. The theorem itself is important, for the theory of similarity depends on it, so that any treatment of Euclidean space, even at the school level, cannot do without the proportionality of segments cut off on transversals by parallels.

However, a proof that presumes a common measure for the <u>entire</u> segments AD and DB, as in Figure 2, is simply incorrect, or at best incomplete. Honesty demands something more, either a rigorous proof or a plain acknowledgement of incompleteness in the proof. For students in the high schools one might go over the matter lightly, showing the truth for the commensurable case and saying we are omitting the problem of incommensurability.

More than honesty is involved; the schools today do talk of irrational real numbers in other contexts, and a thoughtful student who understands the definition and existence of irrationals is bound to wonder about this "proof" concerning triangles, that doesn't take irrationals into account, though it is likely that the average high school student would never even think about incommensurability if it weren't pointed out to him. However the matter is handled by teachers in an actual high school classroom, incommensurability is a matter of such importance in other contexts that the suppressing of all mention of it in the education of a <u>teacher</u> of geometry would be unconscionable.

What that teacher needs is twofold: First, a good mathematical understanding of irrationals and how they impinge on the problem of proportionality in Euclidean geometry; and second, an understanding of the limitations of school children, and some instruction in how to handle this difficult matter in the high school classroom. One should expect a writer of a book on methods of teaching mathematics to have these two qualities himself before seeking to instruct future teachers on the second matter (the pedagogical problem), even if he leaves the first (the mathematical theory of irrationality) to professors of mathematics itself.

In the 19th century there were very few children who completed a high school education, and of these many fewer who had learned much mathematics even as taught in the schools. It was a downright rare child who went through more than Books I and II of Euclid, so that the ancient theory of proportions was generally unknown, and unknown to most teachers as well. Even among professional mathematicians of the year 1900 the theories of Cantor, Dedekind and Weirstrasse were quite recent, and without these the very idea of the irrational in arithmetic, as a strict analogue of the irrational in geometry, or as an explanation for it, was new and strange. The older notion of "limit", so necessary in the differential and integral calculus, could in practice be used, especially after the time of Cauchy (about 1830), without much regard to its philosophical underpinnings, of which the theory of the irrational formed part. And since the Euclidean irrational had been found to be such a stumbling block in Euclidean geometry in the schools, it came to be considered sufficient to transfer the Cauchy idea of "limit" to geometric situations, to provide what seemed a satisfying basis for what otherwise would need irrational ratios in geometry, even if the student never went on to the uses of "limit" in calculus, which of course only a few did.

But the "theory of limits", as seen in late 19th century algebra books for the schools, was full of subtle errors and omissions. Unlike the intuitive example given above, concerning a parallel to the base of a triangle, the algebra textbooks of 1880 or 1900 tended to foster a mystique concerning such things as "variables" and "limits" that would daunt anyone who didn't already know what was behind it all.Every mathematician today recognizes that one cannot avoid Euclid's definition of the equality of ratios, or its Dedekind equivalence in the completion of the rational number system to form the reals, in presenting a really complete account of what is at issue. In the case of infinite series and of continuity of functions, the Cauchy criterion of convergence or its equivalent is a necessity, and cannot be had without some troubling constructions on the number line. In the case of any sensible statement concerning areas of regions bounded by curves, and of the lengths of the curves themselves, and of volumes, other sorts of limits are demanded, even to make sense of so common a thing as the relationship between the side and diagonal of a square. But an inspection of early 20th century textbooks reveals only rather pitiful efforts of the authors to evade the difficulties, oftentimes with the thought that by eliding some uncomfortable connections they were making things easier for their students to understand. It is hard not to believe they were in fact deceiving themselves as much as their students, and that they themselves had only a dim notion of what the era of Cauchy to Dedekind had accomplished in reducing the elements of "continuum" analysis to arithmetic simplicities.

J.W.A. Young's influential The Teaching of Mathematics in the Elementary and Secondary School (New York, Longmans, Green & Co., 1927, p327-346) considered this problem of pedagogy, and recommended against teaching the "theory of limits" in the high schools at all, and with good reason. The theory (which is only alluded to above, via the phrase "in the limit" at the crucial moment in the informal proof of proportionality) was, as Young knew, difficult and in fact incomprehensible to students of high school age (given their earlier preparation in American primary schools) even if presented properly. It is even difficult to convey to high school students that the matter is problematic at all. In 1923, not long before Young's advice (which Young had already put forward in 1911), the National Committee on Mathematical Requirements, in its report The Reorganization of Mathematics in Secondary Schools (Mathematics Association of America, 1923, p35) also recommended that the ideas of limit and incommensurable quantities be given only informally as needed, even though the key theorems on proportionality, e.g. the Euclidean theorems concerning parallels cut by transversals, were among those the 1923 Report considered necessary for students to be able to "prove" (p57). The proof a good high school text would give could only have been the informal use of "in the limit" as used above, of course, for anything more would have had to depend, as Young knew, on developments impossible to teach correctly at the high school level.

Many school textbooks of the next generation followed J.W.A. Young's advice, which, it should be noted, was offered by a mathematician who understood the nature of the real number system and the nature of the Euclidean theory of proportions quite fully. He, and the MAA committee of 1923, did not advise leaving these subtleties out of the high school curriculum because they thought they were mathematically unimportant, or because they were ignorant of their nature. The (1923) National Committee on Mathematical Requirements, which advocated informality (at the high school level) concerning these ideas, had been headed by J.W. Young, chairman of the Dartmouth College mathematics department (J.W. Young, a mathematician, is not to be confused with J.W.A. Young cited above, who was a professor at Columbia Teachers College and while quite knowledgeable in mathematics not himself a research mathematician).

The National Committee included, in addition to many experienced teachers and supervisors from both public and private schools, such university professors as E.H. Moore of the University of Chicago, perhaps America's leading mathematician of the time, and David Eugene Smith, of Columbia University's Teachers College, author of a notable history of mathematics as well as much else, on both mathematics and its pedagogy.

Coincidentally, it might be noted that E. H. Moore's name has long been associated with his own very original theory of limits. The so-called "Moore-Smith" limits are needed for functions whose domain is a more general sort of ordered set than the real numbers (or Euclidean line) or the integers, and it later became formulated by John Kelley as the theory of limits of "nets". An equivalent theory was formulated by Bourbaki in France in terms of "filters" and their limits. These ideas are much more abstract than needed in Euclid, or school algebra, but the names, Kelley and Bourbaki, would later figure in the mathematical landscape in the "new math" era, and their theories of limits (1935-1950) might well be taken as proxies for the sort of abstraction that was said to have obsessed the mathematicians of the period immediately following 1950, as they came to consider the reform of school mathematics education.

In general, the National Committee's 1923 Report could hardly have been wiser, especially in that it strongly advocated a high school program emphasizing the notion of "function", which would surely, they realized, be the unifying theme of mathematics for the foreseeable future. Functions, even if not Moore-Smith limits, were readily understandable by high school age students and their teachers, even if the idea was not yet a commonplace of school mathematics. "Limits" were a commonplace, unfortunately, even where they were quite badly misunderstood.

John Harrison Minnick, Dean of the School of Education of the University of Pennsylvania, did not, on the matter of "limits", agree with either of the Youngs. Minnick (1877-1966) was, after a long career as a teacher of mathematics, from one-room schoolhouse on up to supervisor, became Assistant Professor of education (1917-1920) at the University of Pennsylvania and the author of a series of papers (1918-1920) concerning the diagnosis of students' failure to progress in demonstrative geometry. He was also the composer of the "Minnick Geometry Tests", which could be used in practice, each one testing one of the four "abilities" into which he had partitioned geometric skill: (A) The ability to draw a figure for a theorem, (B) The ability to state the hypothesis and conclusion accurately in terms of the figure, (C) The ability to recall additional known facts concerning the figure, and (D) The ability to select from all the available facts those necessary for a proof, and to arrange them so as to arrive at the desired conclusion. [Minnick's tests are described in more detail in the MAA 1923 Report on pages 381-389.]

During the time he was Assistant Professor at the University of Pennsylvania, Minnick earned a PhD there (in 1918) with a thesis upon this geometry work. (In later years most universities considered it unethical to grant doctorates to their own professors, but in 1918 the practice was still common.) Minnick became Professor in 1920, and then, following a sudden series of important, perhaps angry resignations, was appointed to fill the vacant position of Dean, a post he held until 1948 [See W.W. Brickman's Pedagogy, Professionalism, and

Policy, a history of the University of Pennsylvania's school of education, Philadelphia, 1986].

Coincidentally, in 1921 Minnick began a three-year term as the second President of NCTM, the National Council of Teachers of Mathematics, something he must have been elected to before dreaming he would have to bear the weight of the administration of a School of Education. He was extraordinarily active both administratively and educationally; while Dean he taught courses in the Graduate School regularly, edited and wrote for Educational Outlook, the journal of the University of Pennsylvania Graduate School of Education, and ultimately published his "methods" book, Teaching Mathematics in the Secondary School, in 1939. This being quite late in his successful career, the book was clearly a labor of love and not intended as a pot-boiler. It was his only book, apart from an unpublished book of memoirs now filed in the archives of the University of Pennsylvania.

W.W. Brickman [op.cit.] remarks that Minnick had not earlier been a "scholar", like his predecessor. That predecessor, Frank P. Graves, who had been the first Dean of the University of Pennsylvania's school of education, had a PhD (1892) in classics from Boston University, and another from Columbia (in education) in 1912, and had been a professor of history, and of Greek, and the author of scholarly books. This sort of background and scholarly career, standard for an "educator" of the late 19th Century, was a far cry from Minnick's, which represented the new professionalism of Education: a subject in its own right now, though a subject which earlier had been considered a spin-off or corollary of subject-matter scholarship. Minnick's unpublished memoirs, according to Brickman, record that in his early career he too had shared the usual scholars' prejudice against the field of education itself as a scholarly study; but that he came gradually to believe in its intellectual value as well as its more obvious usefulness in the preparation of future teachers.

While Minnick's Teaching Mathematics in The Secondary Schools (New York, Prentice-Hall, 1939], p.249) acknowledged J.W.A. Young's advice concerning limits and irrationals, and that of the National Committee report of 1923, Minnick deliberately did not take it. He knew better, he said; a properly formulated theory of limits, to be useful in high school geometry and elsewhere in school mathematics, wasn't really hard to understand:

In the Chapter entitled Definitions and Axioms, Minnick argued that, at least for the case of "superior" high school students intending to go on to college, and even if they do not understand fully, some part of the theory of limits is advisable. Mathematics is not learned in one trial alone, he wrote, and early introduction to difficult ideas has an ultimate value. For the benefit of nascent teachers of secondary mathematics especially, he therefore included in his book what he counted a sufficient review of 'the theory of limits' along with some typical applications to geometry.

Minnick was anxious not to repeat what he saw as the errors and ambiguities of the kind of language used in the textbooks that had earned the disfavor of J.W.A. Young (1906), and which the critics of the day had pronounced insufficient (as indeed it was), so on p. 228 he wrote,

... understanding should not be sacrificed for the sake of brevity. An extreme case is the following definition of the limit of a variable.

"K is the limit of the variable x if $|x-K| < \varepsilon$ ***where K is a constant and*** ε ***is an arbitrarily small quantity."***

This definition is brief, and for a <u>college</u> [my emphasis, RAR] student it is satisfactory. For the high school senior, it would be better to analyze this definition into its essentials and use them as a definition, although it results in a greater wordiness. Thus, x approaches K as a limit if

1. K is a constant quantity,
2. x can be made to come as near to K as is desired, and
3. when x has come within a certain distance of K, it is impossible by the same process to make it move farther away.

Minnick returns to this definition later (p.255ff), to prove,

If two variables are constantly equal and approach limits, their limits are equal.

This 'theorem' is used in proving the equality of ratios created by a line parallel to a side of a triangle, by showing (as with Fig.2) the theorem is true when the division ratio is commensurable, then showing the divisions are proportional (in the rational sense) for triangles which can be made as nearly the original triangle as desired in that they share sides and vertex with the original triangle and have a base as close to the original base as desired (Fig.3). But rather than content himself with the intuitive comment we used in this connection, Minnick thinks to formalize the proof by careful appeal to the definitions and theorems on limits as he has given them in his own text. That is, knowing the two ratios (rational numbers, now, rather than Euclidean "ratios") AD/DB' and AD/DC' are "constantly equal" variables, each approaching a limit, the limits (AD/DB and AE/EC) must also be equal. (Equal, <u>whatever</u>

they are, for they are no longer rational numbers, and Minnick's discussion of irrational numbers is yet to be seen.)

Some of what Minnick says about limits in general is defensible, or could be defensible if defended by someone who understood it and provided a few definitions: By "x" Minnick evidently means a function of some independent variable t, t being perhaps a positive integer (in the case of sequences) and perhaps the reals in the neighborhood of some point a where x(a) itself need not be defined -- but which he evidently would like to have value K. "By the same process" evidently means "for smaller neighborhoods of a" (or nearer infinity, for t in N). But even thus generously interpreted, with "variable" meaning real-valued functions, or maybe rational-valued function, and "constant quantity" meaning "real number", his definition is too restrictive in that it supposes limits only for monotone functions, and therefore (e.g.) would never allow a limit for t[sin(1/t)] as t-->0.

Just the same, the implied definition suits his examples, where monotonicity does hold, for the base of the approximating triangles in Figure 3, whose side-segments have rational ratio, can be supposed to proceed (monotonically) downward towards the base of the triangle of the theorem, during the "process" Minnick has in mind, and his expanded definition does away with the nonsensical phrase "arbitrarily small quantity", which he himself had invoked with the "definition" he offered a suitable for a college student.

What is missing, even after all these explanations, which are by no means to be found in his text -- and are certainly not explained in the paragraph just now written above, where the problem is summarized rather than settled -- is the existence of the limit itself, i.e. its very meaning. What is K? It appears to be some "ultimate ratio" as construed in the 17th century and quite properly lampooned by Berkeley as "the ghost of a departed quantity." In the case of the triangle with proportionately divided sides, as described in the Euclidean theorem above the limit K is the ratio AD/DB (or maybe AE/EB), a "quantity" entirely undefined theretofore, that definition of a new sort of ratio being the very point of the whole story. It was the very definition of AD : DB, as given in Book V is Euclid's Elements, that constituted a revolution in ancient geometry, and the translation of the same idea to the number system was not made clear until late in the 19th Century. The definitions themselves, both the ancient one for ratios and the modern one for real numbers, are to this day difficult for undergraduate mathematics majors and entirely unknown even to most users of mathematics (engineers, say), let alone the man in the street.

In one of Minnick's examples for limits, the sum of a geometric series $\sum ar^n$ with rational r, the limit (the sum of the series) is also a rational number which he can actually name and show to be the limit, but in his geometric example, the ratio of the segments created in the sides of a triangle by a line parallel to the base, the very word "ratio" lacks definition in the incommensurable case. For his theory he needs to invoke inequalities of the form $|r(t)-K| < e$, with K the limiting ratio, but there is no such K defined. Giving it a name, e.g. AS/SB, does not bring it into existence or make it understood, in the case where it is impossible that both AS and SB be measured as integral multiples of a common length.

In other words, Minnick fails entirely to recognize the problem J.W.A. Young was warning him against, and he thinks in a few words to straighten out for the benefit of teachers of high school students, and for them to propagate, the use of something he himself had not understood to begin with: the ratio of two arbitrarily given segments. His students, the people reading his book, people being prepared by Dean Minnick for a life of high school mathematics teaching, were being taught to replace a reasonable intuitive understanding with something quite meaningless, though impressively labeled, and indeed something they would then have to memorize for examinations rather than internalize, because one cannot really internalize, i.e. understand, nonsense.

J.W.A. Young, having seen that such ignorance was ubiquitous in his time, simply recommended against going into all this in the schools, a wise counsel not heeded by Minnick. Perhaps Young should have explained the reasons for his counsel, but that would have required him to say, if only inferentially, some impolitic things about the mathematical competence of his colleagues in mathematics education. His advice, thus muted, was not taken, and most high school "advanced algebra" books, preparing students for college analytic geometry and calculus, printed definitions for "limit" in much the same way Minnick did, and used these definitions for equally futile "proofs" of what most children either failed entirely to see a reason for, or ignored. In the latter case they would lose points on examinations when unable to parrot the nonsense on demand, and some of them decided mathematics was too difficult for them, and gave it up.

It is evident from Minnick's entire effort at rigor that he believes he and Young differ only in an opinion concerning pedagogy; he deludes himself that he has discovered a pedagogical device Young hadn't thought of, to make things plain to high school students, where (as Minnick did not understand) Young, like all mathematicians of the twentieth century acquainted with the work of Weierstrasse, Cantor and Dedekind, had seen to the root of the problem: that what might appear to be a mere pedagogical problem in teaching Euclidean geometry to 10th grade students was really so rooted in difficult mathematics that it merely had to be avoided, or at best mentioned as a missing link in the argument.

It is not possible to believe that Minnick really understood the difficulty and chose, for pedagogical reasons, to suppress the "minor point" represented by the existence and properties of the thing he uses as a limit, i.e. the irrational ratios. He clearly did not understand; for his ignorance of this very point is evident in other parts of the discussion of irrationals, and as will be seen below, in contexts much less difficult than this one.

On p. 255, just before stating and proving the "fundamental theorem" about "constantly equal" variables having equal limits, Minnick explains with some insistence that 'incommensurable' is a relation between quantities, not a property of a given quantity, a matter on which apparently students were known to him to be confused. What a "quantity" might be is left uncertain, for if by "quantity" he means "number" he has ignored the problem of irrational numbers, and if by "quantity" he means "segment", then "Having no common measure" in terms of integral numbers of subsegments is the correct definition, he insists. So far, so good, at least in geometry; this is Euclid. Next he intends to prove that incommensurable pairs of segments exist. Euclid of course had already done so, twenty-three hundred years earlier, in a famous proof that (also famously) can hardly be improved upon today, but which Minnick all the same imagines he has simplified.

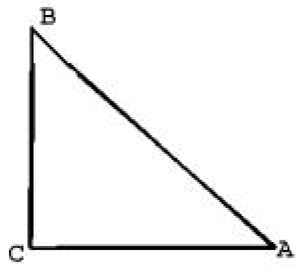

Figure 4

Minnick exhibits (Fig. 4) an isosceles right triangle ABC, with AB the hypotenuse, and posits a common measure for the two equal sides, i.e. a subsegment of AC
of such a length that x of these lengths exactly measure the side. [Such a subsegment, whose length is the (1/x)-th part of the total length of the side (x being a positive integer) can be constructed by a standard Euclidean construction.] A careful calculation with the Pythagorean Theorem (expressed algebraically) then gives "AB = $x\sqrt{2}$." Then Minnick goes on,

"Since it is impossible to find the exact value of $\sqrt{2}$, the chosen quantity is not an exact measure of AB. Therefore, AB and AC do not have a common measure and are incommensurable. Incommensurable quantities are quantities which have no common measure."

Nowhere else in the book is it hinted that the irrationality of $\sqrt{2}$ is problematic; and the phrase "impossible to find the exact value of" wants a bit of explanation, unless it means "not expressible as a fraction", in which case he is being circular. The point of the proof is to show that $x\sqrt{2}$ cannot be an integer, which is identical with the statement that 2 has no rational number for its square root, this being the arithmetic expression for the failure of the side and diagonal to have a common measure. Minnick has restated Euclid's problem in arithmetic terms, and imagines his "impossible to find the exact value of" elucidates something obvious to him. All this in 1939: Euclid forgotten and Dedekind not yet heard of. The students of Professor Minnick were to the time The New Math was to shake them up. Their own students were then also learning, at second hand, and with some ceremony and emphasis too, as befits a mystery, that "there is no exact square root of 2."

Yet Minnick was not a minor figure in mathematics education in his time; his voice was heard from the beginning. In 1916, well before becoming a university professor -- and Dean -- he published in the Mathematics Teacher of December, 1916, a paper, Our Critics and Their Viewpoints, in which he summarize the views of some adverse critics of school mathematics, men who believed that there should be less mathematics in the schools than there was at the time, or a different sort of mathematics. Minnick implicitly defended the current practice of his own time. To the charge that high school students were not in fact learning what they were being offered, as evidenced by a 1/3 failure rate in some New York State examinations, Minnick countered that "by the same standards Italian, Latin, science and the commercial subjects are even worse failures." In particular, he scorned the suggestion that much of high school mathematics cannot be made important or valuable, or even comprehensible, to the average student.

Minnick's was not a majority view in the world of education. One spokesman for the view that mathematics was educationally unimportant, except for a few future technicians, was the famous William Heard Kilpatrick of Columbia University's Teachers College, and while in 1916 the ideas of "progressive education" had not yet taken firm hold of the educational profession, by the time of which we speak (1940, say) mathematics in the schools was of the lowest esteem it would have for the entire 20th Century. [See Cremin, Lawrence A., The Transformation of the School: Progressivism in American Education 1876-1957. NY, Knopf 1961, and Ravitch, Diane, Left Back: A Century of Failed School Reforms, Simon & Schuster, 2000.]

Textbooks for students, in 1940, were even worse than the "methods" textbooks for the teachers, though very few dared broach (as had Minnick) such subtleties as a theory of limits. Subtleties would have made them even worse. In the 19th Century a book with poor exposition was no embarrassment to schoolmaster or pupil, for whatever obscurities the text might contain, the lesson plan was clear: Students were to memorize some announced proof or method, recite it verbatim, then solve a long list of similar exercises demonstrating the use of the formula of the day. That the routine was sometimes incomprehensible made it no different from most of the rest of what had to be memorized, lines from Milton or dates of battles and treaties, which though perhaps comprehensible in principle were not in fact understood by very many of the students. A thoughtful teacher hoped the memorized lessons would be recalled in later life to good effect, and in the case of poetry and oratory they often were, for the student might acquire a vocabulary and an experience of the world over the years that made these memories valuable. Shakespeare and Cicero might not yet mean much to the student memorizing the lines, but the texts did have meaning, which, once rooted in memory, might flower in later years. In mathematics this could only be true if the memorized phrases had meaning to begin with; but Minnick's theory of limits, and much else emanating from the textbook publishers of the time (and, alas, later times), did not.

By 1940 there was in fact a new, 20th Century, "progressive education" atmosphere in the classroom. Recitation *viva voce* was no longer the order of the day. Teachers were to talk with students, hear their ideas, and sympathize with their difficulties; and were forbidden to invoke mere authority to defend their doctrines. All this is quite impossible when the lesson is meaningless and the teacher, naturally thinking it his own fault that he doesn't understand, becomes defensive or evasive to circumvent the students' objections, or questions. In teachers of good will trying their best, the behavior described here might well spring from subliminal fears, or ignorance not even recognized as such, but it was omnipresent among mathematics teachers of the time, the time-servers and the men of good will equally; and the rigidity and defensiveness of teachers more than anything else generated the apprehension among students that they "just weren't good at math." What else can a person say or think, who doesn't understand, however hard he tries, and who is deprived by his own teachers' ignorance of the comfort he might have had from knowing that what he was struggling with merely could not be understood?

Fifty years later, in the latter days of the Soviet Union, there was evident a similar corruption of the spirit, though in Brezhnev's Russia the teacher could be joined by his students in a common hypocrisy when teaching for example the doctrine of the withering away of the State, which nobody believed in any more. To paraphrase a common Russian saying of that time, "We pretend to teach and they pretend to understand." In the 1940 American classroom, with the 1940 textbooks, written by school supervisors and teachers (not mathematicians) who had been educated by a generation of Minnicks and their methods books, the pretense of teaching went on, but without the conscious collusion of the students.

Algebra

The high school level of misunderstanding of mathematics in 1940 extended to even simpler matters than the problem of proportions in Euclidean geometry. It was quite standard for algebra (note: algebra) textbooks at the 9th grade level to include the following carefully stated list of "Axioms":

1. Things equal to the same thing are equal to each other.
2. If equals be added to equals the results are equal.
3. If equals be subtracted...[ditto]
4. If equals be multiplied ..[ditto]
5. If equals be divided...[ditto]
6. Equal powers or roots of equals are equal.

These six axioms are not printed as such in Minnick's 1939 methods book, though he uses the first three freely in their correct geometric context. Those first three are familiar from Euclid, of course, where they have a non-trivial meaning, seldom if ever elucidated in high school geometries. Euclid's idea of "equal" was not identity, as when we say two numbers are "equal" to mean they are the same number; Euclid called two geometric objects "equal" in the first instance if they were congruent, and then if by finite partitions could be matched by pairwise congruence of the pieces, and finally (in Book X of the Elements) if by a process of "exhaustion" they could be approximated by equal figures (equal according to the earlier definition) as closely as desired. (The "approximation" in theorems invoking equal ratios, as for example in the theorem that the areas of circles are to one another as the squares on their diameters, is not a numerical one, and is quite sophisticated, and certainly not a high school sort of lesson. Nor was Euclid's book intended for children at all!)

"Equals subtracted from equals…" also had Euclidean meaning in that (say) congruent figures cut away from larger congruent figures might well yield figures no longer congruent, but equal in Euclid's sense just the same. This notion is at the bottom of the device by which a triangle is shown to be "equal" to half a rectangle having the same base and height, and it is utterly necessary in the very statement of the Pythagorean Theorem, where in no case is it possible to partition the square on the hypotenuse into two smaller squares, but where it is possible to make somewhat finer, though finite, partitions of the three squares in question so that the pieces "add up" properly.

On the other hand, Axioms 4, 5, and 6 can have no general meaning in Euclid, though one might stretch a point in speaking of the product of two lengths as an area. But quotients, powers and roots? To see how ludicrous all six of these "axioms" are one has only to add, e.g.

7. If x = y, then cos (x) = cos (y).

One might as well also "postulate" that the cubes of equals are equal, or their logarithms. These aren't axioms at all, but mere expressions of the fact that $x \dashrightarrow x^3$ and x-->log(x) are well-defined functions. The arithmetical interpretation of Axioms 1, 2, and 3 are equally unremarkable; they are not axioms of arithmetic or logic, but totally unnecessary statements (in their arithmetic interpretation) to the effect that addition and the like have unique meaning, something children had earlier, much earlier, been taught to take for granted, or observe for themselves by counting blocks and measuring table-tops, and were now being urged to mystify. Goodness, what a welter of axioms one could invent in this way, and ask the children to recite as justification when they turn up in a calculation or proof.

Saunders Mac Lane, in The Impact of Modern Mathematics on secondary schools", Bull Nat Assn of 2ndry Sch Principals, XXXVIII (May, 1954, p. 66) and reprinted in The Mathematics Teacher, Feb. 1956, makes this comment: "How many pupils still labor through cumbersome statements like, "if equals are added to the same thing, the results are equal," when they should be dealing with the simpler modern statement: "If a=b, then a+c=b+c." In this paper, written in the early years of the newmath, before Congress began to finance the large projects that characterized the 1960s, Mac Lane, on of America's great mathematicians, was not concentrating on the vacuity of the statement so much as the convenience of modern notation; for it is clear that if "a+c" means anything at all, that is, if it has a well-defined value, the statement is tautological, not something to be taken axiomatically at all. The value of writing the statement as Mac Lane did is in its actual physical use, as a formulation of how a student is to arrange his thoughts in the process of rewriting equations for purposes of solving them, or at least placing their expressions in more convenient written form.

Thus in practice the "axioms" labeled 4, 5, and 6 above are listed in algebraic settings for instrumental use in solving algebraic equations in schools, but from some confusion of mind became conflated with Euclid's quite different ideas into one category. For example, Axioms 2-5

Mathematics, (First Edition, McGraw-Hill 1941) as "the fundamental operational axioms involved in the solution of linear equations," with the admonition, "The student should learn to react without hesitancy to these axioms. They should become so much a part of him that he will come to apply them as readily to literal numbers as to ordinary arithmetical numbers in an equation."

("Literal numbers" meant numbers expressed or denoted for the moment as letters of the alphabet. They are not a new sort of number, even though they are by such notation often also known as "variables". Nor is a number expressed in Roman numerals, or in binary form, a new sort of number. Generations of high school students have been taught by such verbiage as "literal number" and -- more recently -- "variable" to regard the mysteries of mathematics as impenetrable.)

Curiously, the Fourth Edition of Butler and Wren, printed under the same title in 1965, contains the same axioms and the same admonition verbatim on page 326, notwithstanding that the new, improved edition was outfitted with the obligatory opening chapters of "new math" subject matter: truth tables and the axioms for groups and fields, for example. Nowhere in this "new math" material, i.e. in their discussion of group and field in 1965, newly introduced to satisfy the demands of the textbook market of the 1960s, did Butler and Wren find it necessary to observe that if a=b, and if c is also a member of the group, then ac=bc by virtue of Axiom 4 above. Of course, no such "axiom" was needed. In the context of groups, as described in the "newmath" part of the book, something like "ac" is merely taken by virtue of the definition of a group to have a (unique) meaning, in which case if a is b then how can bc become anything other than ac?

But once the Butler and Wren are past all that abstract sort of thing, they forget -- in the same book! -- all about binary relations and groups, which of course are the source of the later supernumerary "axioms" about "equals added to equals", and get back to the business at hand: how to teach the future teachers about equations. Here, as their own teachers had taught them in their own turn, they elevated their misunderstanding of Euclid's axioms into a vast ritual. While it is often noted that The New Math didn't manage to teach its intended message to schoolchildren, we can see here that it didn't even manage to teach its lessons to its own putative purveyors.

To cite actual textbooks predating the "new math" that listed these absurdities would be an endless task; they all did it, none of them recognizing that these facts were redundant versions of what they had --

from kindergarten onward -- been assuming by the mere printing of "a+b", "3+2" and "a/b" as if these expressions meant something. Actually, to anticipate our story a bit, their day is not yet done. For example, these misplaced and misunderstood Euclidean axioms are found -- with elaborations -- in a present-day 9th grade textbook, Integrated Mathematics, Course 1, by Isidore Dressler and Edward P. Keenan (Amsco School Publications, New York, 1980), albeit dressed up in somewhat more modern form, e.g. on page 106:

SOLVING SIMPLE EQUATIONS BY USING DIVISION OR MULTIPLICATION POSTULATES

Postulate 6: Division Property of Equality

***The division property of equality states that for all numbers a, b, and c
(c not equal to 0), If a=b, then a/c=b/c.***

Therefore we can say: If both members of an equality are divided by the same nonzero number, the equality is retained.

As noted above, the form of this statement derives from Euclid, though in the algebra of a field it is quite supernumerary. But Dressler and Keenan don't have axiomatics or logic in mind when listing these things, they have students in mind, students who are to be told what to do when they see an equation. Told to "solve the equation 6x=24" they will dutifully "divide both sides by 6", and have an Answer by virtue of the Postulate. This is all Dressler and Keenan have in mind, and the results are duly certified by the New York Regents Examinations. Everyone involved feels edified by the link with Euclid, or maybe the "axioms of equality" as will be noted below; but the question not answered by Postulate 6 and its relatives is: Both sides of what?

Is "6x=24" an equality? Usually it is not, as for example when x is 10. If we are ignorant of the value x has, is it legal to use Postulate 6? To the apologist who says "Well, x cannot be 10," one must get into a rather lengthier debate than the authors of Postulate 6 bargained for: Is a thing an equation by virtue of the appearance of the symbol "=" in the middle? Is there a difference between an equation and an identity? Why can't you divide both sides of an identity by the same nonzero quantity? (Here the question of what is a “quantity” intrudes; textbooks of the early 20th century and before tended to use the word vaguely, sometimes to mean “number” and sometimes not. In any case, students in high school algebra or trigonometry have always been taught that one does not prove identities by "doing the same thing to both sides".)

One possibly amusing consequence of the confusion of Euclid's "equals" with the "equals" that occurs in algebraic formulas has been an ever more formal elaboration, as the centuries have rolled along, of the properties of the latter usage. Since Euclid's "equals" is an equivalence relation among geometric figures, the property is reflexive, commutative and transitive. The current textbook, Glencoe Algebra I, published (1999) by Glencoe McGraw-Hill and intended for use in 8th grade classes, contains several pages explaining that the symbol "=" as used in such an equation as 3x+5=11 possesses these three properties. Not only may 3x+5=11 be rewritten 3x=6 on the grounds of the Euclidean axiom that equals subtracted from equals are equal, but the original equation may be rewritten 11=3x+5 because of the reflexivity of equality, it is explained by Glencoe.

Again a misreading of something in real mathematics, ignorantly placed in a school textbook because of a chain of ignorance running through a "methods" course in a teachers' college. These three things, identity, reflexivity and transitivity, are true of equality, to be sure, but while they say something worthwhile about less trivial equivalence relations they can do nothing here but confuse the student or make him contemptuous (if he has the courage) of his putative teachers. Both are bad for his education.

What is missing from most current schoolbook accounts of equation solving, in 1999's Glencoe Algebra just as in the typical textbook of 1940, and the typical teachers' college "methods" textbook, is an analysis of what is in fact being stated when one begins with 3x+5=11 and ends with x=2. Such problems today, as in 1940, seldom have any "if" or "then" attached to them as printed in textbooks, and writers who do the extra bit are generally derided as pedantic.

One wonders what the axioms say, and what ritual prescribes, when the equation 2x+5=6-(1-2x) is presented for "solution":

A textbook might (by some hideous error) include the problem,

14. Solve 2x + 5 = 6 - (1 - 2x)

(Notice, among other things, that there is no period at the end of this sentence. Textbook mathematical sentences are not real sentences, and don't need capital letters at one end or periods at the other. Such an example for children's reasoning! Such an example for their prose!)

The dutiful student will then, by the procedure taught him to "solve the equation", write the following sequence of punctuationless lines:

1. 2x + 5 = 6 - (1 - 2x)
2. 2x = 1 - (1 - 2x)
3. 2x = 1 - 1 + 2x
4. 2x = 2x
5. 0 = 0

and then wonder what happened. Has he solved the equation? He has read in a textbook (I have seen it somewhere) that "to solve an equation is to isolate the unknown on one side of the equation with a numerical value on the other." Whether he has read that instruction explicitly or not, that is surely what his book and teacher expected, and they expected him to use the "axioms" of equality in the process. Well, yes, he has done so. In the first step he subtracted 5 from both sides. In the next step he "removed parentheses", the rule being, when the parenthesis is preceded by a "minus" sign, to obliterate the minus sign and both parentheses, rewriting the terms within the parentheses with changed signs, including if necessary the "+" sign for the first term if that term had earlier, i.e., within the parentheses, been afflicted with a "-" to its left. He might even have learned why this gives the correct result, though the confusion of the meanings of the minus sign, which sometimes indicates the negative of a number and sometimes the subtraction of that number, according to the syntax, made it almost impossible to explain to children why line 2 in the sequence listed above becomes line 3 via the distributive law. From Line 3 to line 4 is easy, and then "0=0" is plainly the result of "subtracting the same thing from both sides".

During the days of The New Math the rule for "removal of parentheses" did undergo explanation, to the derision of some detractors who thought it belabored the obvious. To explain the rule, which is quite complicated as written above, it is necessary to have a notation for negatives (additive inverses) -- a superscripted dash or wiggly hyphen, say -- distinguishable from the notation for subtraction, and these were in fact introduced in many of the new programs, though explanations generally followed fairly soon, making the extra notation unnecessary and restoring the traditional uses of the dash as above. For example, if "~2" denotes the additive inverse of 2, it is a <u>theorem</u> that "5+(~2)" and "5-2" represent the same number. Only after having proved such theorems in general can the customary notations and rules concerning parentheses and signs become reliable. Even so, the derision remained, one among many derisions elicited by the new math in the textbooks.

This is not to say that children in this or that grade when first learning to manipulate algebraic expressions should be subjected to all the theorems and proofs of elementary group theory, only that there are many things, and in 1940 these were legion, that were simply not understood by the professors of education and the textbooks they wrote. Today's mathematically educated reader can trace these misunderstandings through to the sadly inadequate treatments of algebra, and even elementary arithmetic, that were therefore offered in the schools. It is not necessary for a teacher to burden the student with the whole truth, but a teacher who misconceives this truth is bound to make mistakes, or talk nonsense, in some contexts. A student should know the status of his knowledge, even if he has to be told that he will know the reasons for some of what he does only when he is older. Do we not use the same principle when teaching history, or the literature of love and death?

Yet well before SMSG and the omnipresence of "new math" in the schools, Saunders Mac Lane's article, The impact of modern mathematics on secondary schools, in MT, February, 1956, p66-69, noted that "...proof is the form in which all mathematics appears, be it geometry or algebra or calculus." It was a bit later in the same article (p67) that he wrote what was quoted above concerning the "axioms" as used in solving equations, "equals added to equals" and all that. In all this he was quite correct, though apparently not prescient enough to see what would become of "proof" when the wide world, much of it un-comprehending despite -- as in the case of Minnick -- its certification as mathematics educators, attempted to enforce the language of proof upon students and their teachers.

Let us suppose now that the student does understand what is involved in "removing parentheses", "transposing terms", and "changing signs", by using the properties of the field of rational numbers, as most high school algebra students can do even if they haven't been taught about fields as such. Our student still comes up with "0 = 0" where he had expected a "solution" to the equation. Where did he really go wrong in the case of 2x+5=6-(1-2x)? Mac Lane's brief statement supplies the answer.

Each of the five lines of the student "solution" is a sentence, or clause in a longer statement, with "=" the verb in each case. The lack of capital letters and punctuation customary in school textbooks so obscures this truth that there appears to be no place in this "solution" for Mac Lane's "simpler modern statement": "If a=b, then a+c=b+c." But it is there. The first two lines

$$
\begin{aligned}
2x + 5 &= 6 - (1 - 2x)\\
2x \quad &= 1 - (1 - 2x)
\end{aligned}
$$

of the displayed solution are really one sentence, when the punctuation is put in:

"If 2x+5 = 6-(1-2x), then 2x = 1-(1-2x),"

2x+5 being Mac Lane's "a", 6-(1-2x) his "b", and -5 his "c".

Even so, there is nothing self-explanatory about all this, for the "statement" contains a symbol, x, to which we have not yet been introduced. A mathematical statement, like any other statement, has to be about <u>something</u>, and this one seems to be about "x"; but who is x? It is as if we were writing a biography using only "he" and "him" whenever it came to mentioning the person whose life is described. The sentences would be grammatical, and some logical ones plainly correct, but none of it would have meaning. For example, the biography might begin, "Born in January of 1843, he was fourteen years old when he entered Cambridge in 1857." This could be true, in that 1857 – 1843 = 14, but it is not yet much in the way of biography, until we know who "he" was, or at least that there ever was such a person. The statement might well have said, "Born in January of 1855, he was two years old when he entered Cambridge in 1857", and would be equally (logically) undeniable, though now we might well suspect that the "he" of this particular sentence probably did not exist.

Mac Lane was himself being elliptical in his observation about adding the same thing to equals, and to be more complete should have written, "<u>If a, b, and c are numbers</u>, and <u>if</u> a=b, then a+c=b+c." Without the "quantifiers", the advance announcement that a, b, and c are postulated as numbers, the statements that follow have no referents; they concern pronouns, not things, not even hypothetical things. In the case of the equation 2x + 5 = 6 - (1 - 2x), Mac Lane's dictum prescribes, as a first step in the attempted solution, "If there is some number x such that 2x + 5 = 6 - (1 - 2x), then 2x = 1 - (1- 2x)," and this is both meaningful and correct. Whether it is of any value, however, is another question, which will be considered below.

It is customary, though unfortunate, that the quantification "Suppose x is a number .." is so often elided in setting problems of this sort in textbooks. One might say, "Well, of course they are numbers we're talking about; why make an issue of saying it all the time?" That there is an answer to this will appear when we have done analyzing "what went wrong" in the conclusion "$0 = 0$" that so puzzlingly emerged from the routine prescribed by high school algebra in the case of the equation, $2x + 5 = 6 - (1 - x)$.

Now in school algebra, "axioms" (or "postulates") such as those mentioned in the Butler and Wren textbooks, have, apart from their Euclidean ***cachet***, a second attraction, which is why they continue to be repeated, however nonsensically, to the present day. They provide a prescription, almost an algorithm, for "solving equations". From $x+6=15$ we get $x=9$; why? Today's school algebra book is proud of itself for not saying, as was said fifty years ago, "because you can transpose the "6" to the other side of the equation with a change of sign." No, that would smack of authority and the mindless memorization of a Rule. Instead, it is said: "Subtract 6 from both sides of the equality and the result is also equality, by Axiom 3."

The method works, of course, and because it seems to lean on an Axiom rather than a memorized ritual named "transposing" it is considered advanced and rigorous. But in truth, the reason "$x=9$" answers the question is not this at all: 9 is a solution of the problem "Find a number x such that $x+6=15$" because $9+6=15$. That's all. In most school algebra books it is not even made plain that this is the proof of the solution. The exercises begin, "Solve the equation ....", and the application of a suitable number of instances of the axiom list produces a new equation with "x" isolated on the left and some number on the right; this number is called the solution of the equation, and the observation that $9+6=15$ is generally construed as a "check", a test of whether someone has made a numerical error. But the "check" is in fact the proof of the solution's validity, and the earlier part was only exploratory.

For, to continue the simple case of "$x+6=15$", what has the "axiom" about subtracting 6 from both sides told us? Only that if there is some number x such that $x+6=15$, then x must be 9. The tedium of expanding the verbiage to include the "if" and the "then" does indeed tell us something the bare repetition of a list of equations derived from one another by applying certain rules does not. It does not tell us that 9 is a solution of the equation; it does not even address that question. It only tells us that if there is a solution, that solution can only be 9. Whether 9 is or is not a solution must be tested separately. Fortunately it tests well and fortunately for thoughtless writers of school algebra books results arrived at in this way usually test well because the equations offered for solution are of a particularly simple type. (The mischief wrought by the usual ritual does not ordinarily appear until quadratic equations are encountered.) In general, however, using the so-called axioms produces only a uniqueness proof: that if there is a solution (in the linear case) there is only that one. Only substitution assures us that number does the job, i.e. that the number substituted for x in the original equation is indeed a solution. This latter step does not say the x in question is the only one (though in the present case it is). Both parts of the “solution” to the problem are needed if the answer is to be definitive.

To emphasize this point, one might apply the axioms in the following way, the way I have called "exploratory", to the equation $x+6=15$. Let us apply Axiom 3, "When equals are multiplied by equals, the results are equal", as before. Very well; assuming there is an x such that the equation is indeed an equality, let us multiply both sides by $(x-1)$. We obtain $x^2+5x-6=15x-15$, from which by the usual rules we get $x^2-10x+9=0$, which factors into $(x-9)(x-1)=0$, and we have two answers: $x=9$, and $x=1$. Is there a mistake somewhere? Where did that extra answer, 1, come from? It certainly does not satisfy the equation $x+6=15$. Is the axiom incorrect, about multiplying both sides of an equation? Not at all, everything done here is correct even though school teachers and textbooks frown on “multiplying both sides” by an expression “containing the unknown”.

Well, they may be right to frown as a practical matter, for a thoughtless student applying the axioms and keeping no account of the logic of what he is doing will “get two answers”, one of them false in this case. If this routine is all that is being taught under the heading of “equation-solving methods”, the teacher is right to warn against multiplication by $x-9$, and make special rules about using the “multiplication axiom” in such cases. But does this mean the “Axiom” is false for some numbers (x, and hence $x-9$, was assumed to be a number, was it not) and true for others? Textbooks for a century or more have been ambiguous about such things, and in the 19th Century developed a mystique about “extraneous solutions” that grew so incomprehensible that by 1940 the standard school algebra book simply didn’t try to explain it, and instead confined itself to ***ad hoc*** prohibitions.

Squaring both sides of an equation was another popular no-no.

Yet there was never anything wrong with applying the rule about multiplying both sides of an equation by the same thing; there couldn't be, since doing so amounts to the mere (true) statement that multiplication by a number has a meaning, a unique meaning. What appears wrong in the case of getting an extraneous root is really a misreading of what the sequence of "steps" in the "solution" is really saying. Taken all together, the process outlined above, where (x-9) is multiplied into both sides, produces the statement, "If x is a number such that x + 6 = 15, ...(etc.), then x must be either 9 or 1." This is perfectly true. It is like saying, "If Jack lives in Ishpeming, Michigan, he lives in either the United States or in China." There are those who don't like this use of the word "or", but it is used this way in mathematics because there is hardly any other way to express the idea, especially when one is for the moment unable to verify which of the alternatives is the case. One might imagine a resident of Tibet who has never heard of Ishpeming or Michigan, but does know about China and the USA. The statement that "Ishpeming, Michigan" is in one of the two named countries is real information for this person, and the fact that with study (or an atlas, say) one can narrow it down further does not mean that the statement as given is incorrect, or even uninformative.

Thus, while multiplying both sides of x + 6 = 15 by (x-1) was not useful, as it turned out when "checking" the candidates for solution, it did not lead to error. It is just that "if…, then …" is not always a two-way street, and the process of solving equations by "doing the same thing to both sides" can convey falsehood only to those who cannot understand the meaning of implication.

But this careful delineation of the "if…, then…" character of algebraic reasoning is more than mere philosophy, designed to render complicated the simple process, taught in Grade 8 or so, of solving a simple equation. Let us return to the rather mystifying "0 = 0" that emerged when "Solve (3x+2) + 7 - (2x+1) = x+8" was attacked by the methods usually given for solving equations. Perhaps we were assured by our teacher that when linear equations were in question, and we didn't square both sides, or multiply both sides by a quantity containing the unknown, we would stay out of trouble. And as earlier described we keep to the rules and still get in trouble: the problem reduces to "0 = 0".

There may be teachers who say, "If you end up with 0 = 0, that means any x will solve the equation." They are mistaken, since one can end with 0 = 0 by multiplying both sides of 3x = 6 by zero (Axiom 4 above!), yet "any x will solve it" is false; there is only one solution to this equation. The germ of truth in the teacher's interpretation of the meaning of 0 = 0 is that it -- the assertion that any x will solve the equation -- is true if the operations on the equations are all invertible; but to show this one would have to go further into the logic of the solution of equations (and the proving of identities, an even more vexed school algebra or trigonometry ritual of the time) than can be squeezed out of the six axioms for equality, such as they are.

The analysis given here is simpler. Our procedure has in fact succeeded in giving us a correct statement of implication, i.e., that if x is a number such that $(3x+2) + 7 - (2x+1) = x+8$, then $0 = 0$. All quite true and quite useless except in that it alerts us to the fact that our procedure in trying to find a solution is not telling us anything, and that we had better try some other way. This could be useful information, even if it does not lead to what we expected.

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The founders of The New Math of the 1950s recognized these lapses in school mathematics immediately on looking into a sample textbook. It is today worthwhile to ponder what a shock a school mathematics book was to a practicing, adult mathematician when it was brought to his attention in 1950, for we all (those of us who were not immigrants) had studied from such textbooks in our own childhood. How quickly we had forgotten our own beginnings!

The hiatus in life occasioned by the World War II undoubtedly had much to do both with our shock and with the form taken by our desire to do something about it. My own history was fairly typical. I graduated from a public high school in 1941 and spent a year and a half in college before entering the Army, where I became a radar maintenance officer. The mathematics I learned in high school was partly useful, and I learned to pass all the examinations very well. Euclidean geometry was the best taught, because the textbook was meticulous and we had to work out a lot of proofs and constructions of our own. The algebra was symbol manipulation, but nobody tried to tell us about limits. The trigonometry had a week or two of information, and the rest had to do with interpolation in tables of logarithms, correct as far as it went but more designed for an 18th Century surveyor than for the electronic future. I didn't think much about the philosophic basis for anything I learned, and while some of my high school chemistry was illuminating the physics was at best a couple of experiments of Archimedes and Thomas Hooke, the lever and Young's Constant.

College algebra and analytic geometry, and calculus, opened a new world. There were limits and real numbers, to be sure, but nobody made a great point of proving calculus theorems rigorously, and our textbooks, written by mathematicians, didn't affect to uncover the mysteries of Dedekind and Cantor, and so didn't stupefy us with false notions. It is true that we didn't quite understand why $(-a)(-b) = ab$ for real numbers, but we didn't much think about it. During the war I learned a lot about radio and radar, and came back to college in 1946 to major in physics. As a graduate student I turned to mathematics, and wrote a thesis on an arcane question about linear topological spaces. In the meantime I had read some of the earlier volumes of Bourbaki, and had come to a good appreciation of the axiomatic basis and logical construction of much that I had merely taken for granted when in school, things I hadn't known before as well as things I had “sort of” understood. I forgot, through all this, that I had been downright lied to in school mathematics courses, since it didn't seem to me that anything might have been different. You learn more as you grow older.

But graduate school had been a sort of cocoon. In 1952 I began teaching at the University of Rochester, and was shocked to see how little my freshmen students seemed to know or understand. They insisted, for example, that π was not an exact number, and that $\sqrt{2}$ was the name of two numbers simultaneously. Despite being able to recite the Pythagorean Theorem, they could also believe that $(a+b)^2 = a^2+b^2$, or at least behave as if they believed it. They could not be made to understand the process of proving a theorem by mathematical induction, and some of them were convinced the thing was a fraud, assuming the result in order to prove it. Fundamentally, they (or many of them) simply didn't believe that mathematics was written in English, or was designed to be anything but a tool, like an automobile or radio, which one learned to operate step-by-step without worrying about what was inside. That we professors of math sometimes refused to tell them the next step struck them as unfair. The things we were asking on examinations were, by their lights, "trick questions".

Not all this happened in every class with every student, and in fact the earlier students, those I and my generation had taught when we ourselves were graduate students and teaching assistants, were themselves, in 1948 and 1949, veterans of the war, "GI Bill students", more adult than the usual run of college freshmen and more amenable to instruction, but by the time I came to Rochester in 1952 the juveniles were back, and the kind of school education they had been receiving was painfully visible. It could hardly have been worse, since in the war years and immediate aftermath, people skilled in anything scientific, physics or chemistry or mathematics, were taken from the schools into industry or the military, leaving behind, as teachers, the least qualified. The postwar inflation then left teachers' salaries far behind, public inertia in such matters being what it is. And the textbooks all dated from the time of Minnick, or were written by educators taught by professors of Minnick's era and outlook.

In 1955-56 I spent a postdoctoral year at Yale University, only fifteen years away from my own high school math books. One of my young colleagues found an old (perhaps 1930) school math education book in a sidewalk rack book sale, and bought it for five or ten cents to show the rest of us. Its idiocies were everywhere dense and we all made merry over its statements, statements intended, like those referred to above, to make clear to teachers what the truth was that lay behind what they would be teaching children about arithmetic. Funny! We were young, and reform was not yet really in the air, for all that some professional educators were already trying to do something about it. Beberman in Illinois, for example, was not yet known to us. Nor did we know that just two years later, right there at Yale University, there would begin the greatest assault on mathematical illiteracy the country had ever known.

I don't remember much of what the book said, but only that it provided us with a couple of phrases we were able to use for the rest of the year, to characterize, or refer to, the gulf between school arithmetic and our own understanding of mathematics. In particular, the author had solemnly catalogued the separate pieces of arithmetic information that should be known to children by Grade 2, Grade 3, and so on, under the rubrics, "Addition Facts", "Subtraction Facts", "Multiplication Facts" and "Division Facts". For example, "8 - 3 = 5" is a subtraction fact.

There is nothing wrong with such terminology, one supposes, provided there is some use made of it. The author never hesitated. My recollection is that he showed that forty multiplication facts (maybe the number was more than this, or fewer) were sufficient instant knowledge for all multiplication, once one learned the "long multiplication" algorithm, and maybe a few other short-cuts, such as (the fact!) that 1XK=K for every K, and that AB=BA for all A and B. Not that he used such symbolism, of course. Well, this too is true. What caused us the greatest merriment was that he totted up all the facts and produced a grand total of "arithmetic facts" children should have learned by a certain time, or maybe several such stages of development. (I have since found that research in the psychology of education has taken the counting of important facts seriously, one researcher in a volume published by Columbia's Teachers College, having found that since ½, 1/3, 2/3, ¼, 1/8, and 1/16 accounted for 99 % of all fraction usage in American industry and banking – he had made a survey! – there was little reason to trouble youngsters with other, unpleasant sorts of fractions when they were being taught arithmetic.

As I say, we were young; and merriment comes more easily to the young than to the old. But if it was not funny, it was sad. Here was a book designed to teach potential teachers of arithmetic, and instead of teaching its subject it wasted its space, and the readers' attention, on <u>counting</u> the facts to be learned by small children by the end of Grade 4. This does not rank high as a crime against the world of learning, of course, but it stands in my memory as an introduction, two or three years before I had heard of "the new math", to the world of math education that the reformers of the 1960s were out to change.

Men whose research in education drives them to write such stuff will never understand the Eudoxus or Dedekind definitions of the irrational; this goes without saying. But worse, behind the fatuity of the counting of division facts there was an even greater void: the misunderstanding of the nature of even the simplest system of numbers, the positive integers themselves.

Young as we were, there in New Haven in 1955, we had all had some experience in teaching college students. Certain experiences were common to us all: the difficulty of teaching "mathematical induction" being one of them. What did they lack, these students who considered the argument circular because it began with the apparent conclusion: "Suppose the proposition true for the case n"? Were they lacking in "addition facts", these students, or of algorithms for "long addition"? Had they forgotten some of the six axioms for equations?

Not at all; they had simply never been taught the most important "fact" of all: the nature of the system of reasoning as a whole, and in particular the meaning of a hypothetical statement from which a deduction may proceed. Had they learned what they were doing when solving x+6=9 they would have been able to follow the later, more intricate, reasoning involved in mathematical induction.

A second experience was recognized by all of us (and this, like the inability to understand mathematical induction in N, is recognizable to every freshman calculus teacher even today, forty years later). This experience is taken from the first week of a college calculus course where it is customary to explain, with numerical example, the meaning of "derivative":

We explain at the blackboard, with a graph and much conversation with the class, how one can find the slope of a line tangent to the curve $y=x^2$ at the point (2,4). How? We draw a line through that point and label its other intersection with the curve by (2+h, 4+k); we compute k from knowledge of h and the fact that $y=x^2$ at all points, and thus get the slope of that chord; we catalogue the slopes for various values of h, positive and negative, large and small, in a table of two columns, the "h" column and the "Slope(h)" column, we notice that for small h there seems to be a condensation of results -- well, no need to repeat the whole story here, we get our result: the limiting slope is 4. We are careful, at first, not to generalize to a point (a,a^2), nor to offer too abstract a definition of "limit". One thing at a time.

We repeat the process for another point, perhaps, (-3,9). With due care we go for a "general" point (a,a^2). With some trepidation we rename that point (x,x^2). We conclude with a formula: If $y = x^2$, then $y' = 2x$. We review what we have done. We feel fine. Any questions?

"Yes," says the kid in Row 3. "Why did you have to do all that, when you could have done it the easy way?"

"The easy way?"

"Yes. You take the exponent and make it the coefficient, and then you reduce the exponent by one."

I was stunned when I heard this for the first time; later I got used to it. What should be the answer? I believe I said, "But how did you know that would produce the answer?" And the kid said, "Because I learned it in our calculus course in high school."

Such a student is hard to teach. He has already come to believe that mathematics is a list of facts and procedures unrelated by logic. He might even understand the meaning of "derivative", as he understands the meaning of "subtraction", but he regards the news that the derivative of x^2 is 2x a mere fact like the news that 8X7=56. Somewhere in his past he learned such things without his own intervention. In the case of 8X7 he perhaps could prove the result with a diagram, but somewhere along the line, when words like "variable", "limit" and "logarithm" were bruited about in his math class, his logical faculties tuned out, and the rest of what he learned was by catalogue, nor did he remember that there could be a logical path from earliest number sense to what he thought he knew now.

There, at Yale in 1955, we all could see the root of this blindness in the counting of division facts by the faculties of the education colleges, and the solving of equations by means of nonsensical axioms, with the logic of solution turned inside-out. We all knew, without even holding a conference sponsored by the NSF, that the only cure was a decent definition of number and figure, and a decent attention to the logical structure of what we were saying. To us, who had written papers for meetings of the American Mathematical Society, for whom the slightest failure of logic negated all, this seemed like very little to ask. Little as it was, it was surely no littler than the minimum that every citizen should have at his command. So much we could not help but believe, those of us who had never actually dealt with the world of pupils in the schools, and their teachers and teachers' teachers, and those who publish books for that enterprise. We could believe it was little to ask because, among other things, we had never read Minnick's methods text, and we had never seen a college course in mathematics education.

When, a few years later, the National Science Foundation began paying hordes of real mathematicians to prescribe for the schools, it was inevitable what, in our own form of innocence, the mathematicians would prescribe. Once numbers were understood as forming an ordered field, and the positive integers among them as a certain inductive subset, and once the language of sets became standard, so that statements with quantifiers made sense, then and only then would students see that the derivative of the square function was not a mere fact in a catalogue but an idea within their own power to reproduce. Only with such power, after all, could any real good come of such knowledge.

And that isn't hard, actually (so it seemed to us). Let us at least try, and test out the results on schoolchildren, to see if we are going too fast or unnecessarily slow. Let us also make sure the teachers who are to teach these essential preliminaries themselves understand what they are doing. And for once we can have books that tell the whole truth and nothing but the truth, so that teacher and pupil alike will be able to fall back on something valid, and not have to participate in a charade of pretending to teach and pretending to learn.

The rest is history, a sad history, which will have to wait for another chapter.

Ralph A. Raimi

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