Tropical Varieties

Jan Verschelde

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
`http://www.math.uic.edu/~jan`
`jan@math.uic.edu`

Graduate Computational Algebraic Geometry Seminar

Tropical Varieties

1. Introduction
   - Introduction to Tropical Geometry
2. Hypersurfaces
   - tropical varieties defined by one polynomial
   - tropical varieties and skeletons
3. The Fundamental Theorem
   - tropicalization of a variety
   - steps in the proof
4. Multiplicities and the Balancing Condition
   - assigning multiplicities to rays
   - balancing a fan
   - the structure theorem

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Introduction to Tropical Geometry

Introduction to Tropical Geometry is the title of a forthcoming book of Diane Maclagan and Bernd Sturmfels.

The web page
`http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.html`
offers the pdf file of a book, dated 31 March 2014.

Today we look at tropical varieties.

This seminar is based on Chapter 3.

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tropicalization

$K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the ring of Laurent polynomials over K.
For $f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ the *tropicalization of f* is a piecewise linear concave function

$$\mathrm{trop}(f)(\mathbf{w}) : \mathbb{R}^n \rightarrow \mathbb{R} : \mathbf{w} \mapsto \min_{\mathbf{a} \in A} \left(\mathrm{val}(c_{\mathbf{a}}) + \langle \mathbf{a}, \mathbf{w} \rangle \right).$$

The classical variety of f is a hypersurface in the algebraic torus T^n over the algebraically closed field K: $V(f) = \{ \mathbf{z} \in T^n : f(\mathbf{z}) = 0 \}$.

Definition

The *tropical hypersurface* $\mathrm{trop}(V(f))$ is the set

$$\{ \mathbf{w} \in \mathbb{R}^n : \text{ the minimum in } \mathrm{trop}(f) \text{ is achieved at least twice } \}.$$

Let $V(F) = \{ \mathbf{w} \in \mathbb{R}^n : \text{ the minimum in } F \text{ is achieved at least twice } \}$ for a tropical polynomial F, then $\mathrm{trop}(V(f)) = V(\mathrm{trop}(f))$.

the fundamental theorem for tropical hypersurfaces

Theorem (Kapranov's Theorem)

For $f \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$*, the following three sets coincide:*

1. *the tropical hypersurface* $\mathrm{trop}(V(f))$ *in* $\mathbb{R}^n$*;*
2. *the closure in* $\mathbb{R}^n$ *of* $\{ \mathbf{w} \in \Gamma_{\mathrm{val}}^n : \mathrm{in}_{\mathbf{w}}(f)$ *is not a monomial* $\}$*;*
3. *the closure in* $\mathbb{R}^n$ *of* $\{ (\mathrm{val}(z_1), \mathrm{val}(z_2), \ldots, \mathrm{val}(z_n)) : \mathbf{z} \in V(f) \}$*.*

In addition, if $\mathbf{w} = \mathrm{val}(\mathbf{z})$ *for* $\mathbf{z} \in (K^*)^n$ *with* $f(\mathbf{z}) = 0$ *and* $n > 1$*,*

then $\{ \mathbf{y} \in V(f) : \mathrm{val}(\mathbf{y}) = \mathbf{w} \}$ *is an infinite subset of* $V(f)$*.*

This theorem will serve as the base case for the fundamental theorem.

lifting zeroes of initial forms

Proposition 3.1.5 is used to prove Kapranov's Theorem.

Every zero of an initial form of f lifts to a zero of f.

Proposition (Proposition 3.1.5)

Let $f \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

- *Let* $\mathbf{w} \in \Gamma_{\rm val}^n$ *for which* $\mathrm{in}_{\mathbf{w}}(f)$ *is not a monomial.*
- *Let* $\mathbf{z} \in (\mathbb{K}^*)^n$ *satisfy* $\mathrm{in}_{\mathbf{w}}(f)(\mathbf{z}) = 0$.

There exists a $\mathbf{y} \in (K^*)^n$: $f(\mathbf{y}) = 0$, $\mathrm{val}(\mathbf{y}) = \mathbf{w}$ *and* $\overline{t^{-\mathbf{w}}\mathbf{y}} = \mathbf{z}$.

If $n > 1$, *then there are infinitely many such* $\mathbf{y}$.

The proposition is reminiscent of Hensel's Lemma.

tropical varieties and skeletons

A *k-skeleton* of a polytope is the union of its k-dimensional faces.

Proposition (Proposition 3.1.6)

Let $f \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. The tropical hypersurface $\mathrm{trop}(V(f))$ is the support of a pure Γ_{val}-rational polyhedral complex of dimension $n-1$ in $\mathbb{R}^n$. It is the $(n-1)$-skeleton of the polyhedral complex dual to a regular subdivision of the Newton polytope of $f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ given by the weights $\mathrm{val}(c_{\mathbf{a}})$ on the lattice points in A.

The coarsest polyhedral complex such that $\mathrm{trop}(f)$ is linear on each cell is denoted by $\Sigma_{\mathrm{trop}(f)}$. The maximal cells of $\Sigma_{\mathrm{trop}(f)}$ have the form

$$\sigma = \{ \mathbf{w} \in \mathbb{R}^{n+1} : \mathrm{trop}(f)(\mathbf{w}) = c + \langle \mathbf{w}, \mathbf{a} \rangle \},$$

where $c \odot \mathbf{x}^{\mathbf{a}}$ runs over the monomials of $\mathrm{trop}(f)$. $|\Sigma_{\mathrm{trop}(f)}| = \mathbb{R}^{n+1}$.

polyhedral complex induced by valuation

Proof of Proposition 3.1.6. $f = \sum_{\mathbf{a} \in A} c_\mathbf{a} \mathbf{x}^\mathbf{a}$ has Newton polytope

$P = \mathrm{conv}(A)$ and $P_{\rm val} = \{ \, (\mathbf{a}, \mathrm{val}(c_\mathbf{a})) \mid \mathbf{a} \in A \, \}$.

A lower face $\mathrm{face}_\mathbf{v}(P_{\rm val})$ of $P_{\rm val}$ is determined by a $\mathbf{v} \neq \mathbf{0}$:

$$\mathrm{face}_\mathbf{v}(P) = \{ \, \mathbf{x} \in P_{\rm val} : \langle \mathbf{x}, \mathbf{v} \rangle \leq \langle \mathbf{y}, \mathbf{v} \rangle, \text{ for all } \mathbf{y} \in P_{\rm val} \, \}.$$

Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the first n coordinates. The regular subdivision of P induced by $\mathrm{val}(c_\mathbf{a})$ consists of all $\pi(F)$, for F ranging over all lower faces of $P_{\rm val}$.

$\mathcal{N}(F) = \{ \, \mathbf{v} : \mathrm{face}_\mathbf{v}(P_{\rm val}) = F \, \}$ is the normal cone of F.
$\widetilde{\pi}(\mathcal{N}(F)) = \{ \, \mathbf{w} \in \mathbb{R}^n : (\mathbf{w}, 1) \in \mathcal{N}(F) \, \}$ is the restricted projection.

The collection of all $\widetilde{\pi}(\mathcal{N}(F))$ as F ranges over all lower faces of $P_{\rm val}$ forms a polyhedral complex in $\mathbb{R}^n$ that is dual to the regular subdivision of P induced by $\mathrm{val}(c_\mathbf{a})$.

$\mathrm{trop}(V(f))$ is the $(n-1)$-skeleton of $\Sigma_{\mathrm{trop}(f)}$

Proof continued. If $(v_1, v_2, \ldots, v_n, 1) \in \mathcal{N}(F)$, then $\mathrm{in}_{\mathbf{v}}(f)$ is supported on $\pi(F)$ and $\pi(F)$ is the Newton polytope of $\mathrm{in}_{\mathbf{v}}(f)$.

This means: $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathrm{trop}(V(f))$ if and only if $\mathbf{w} \in \widetilde{\pi}(F)$ for some face F of P_{val} that has more than one vertex.

So $\mathbf{w} \in \mathrm{trop}(V(f))$ if and only if $F = \mathrm{face}_{(\mathbf{w},1)}(P_{\mathrm{val}})$ is not a vertex.

This happens if and only if the face $\widetilde{\pi}(\mathcal{N}(F))$ of the dual complex that contains $\mathbf{w}$ is not full dimensional.

We conclude: $\mathrm{trop}(V(f))$ is the $(n-1)$-skeleton of the dual complex, and this is a pure Γ_{val}-rational polyhedral complex. □

an important case

In case the valuations of the coefficients of f are all zero, the tropical hypersurface is a fan in $\mathbb{R}^n$.

Proposition (Proposition 3.1.10)

Let $f \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial with coefficients that all have zero valuation. The tropical hypersurface $\mathrm{trop}(V(f))$ is the support of an $(n-1)$-dimensional polyhedral fan in $\mathbb{R}^n$. That fan is the $(n-1)$-skeleton of the normal fan to Newton polytope of f.

The complex $\Sigma_{\mathrm{trop}(f)}$ is the normal fan of the Newton polytope of f and we apply Proposition 3.1.6.

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tropicalization of a variety

Definition

Let I be an ideal in $K[\mathbf{x}^{\pm 1}]$ and
let $X = V(I)$ be its variety in the algebraic torus T^n.

The *tropicalization* $\mathrm{trop}(X)$ *of the variety* X is the intersection of all tropical hypersurfaces defined by Laurent polynomials in the ideal:

$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f)) \subseteq \mathbb{R}^n.$$

By a *tropical variety* in $\mathbb{R}^n$ we mean any subset of the form $\mathrm{trop}(X)$ where X is a subvariety of the torus T^n over a field K with valuation.

A finite intersection of tropical hypersurfaces is a *tropical prevariety*.

tropical basis and tropical variety

A finite generating set $\mathcal{T}$ of I is a tropical basis if for all $\mathbf{w} \in \Gamma_{\rm val}^n$, $\mathrm{in}_{\mathbf{w}}(I)$ contains a unit $\Leftrightarrow \mathrm{in}_{\mathbf{w}}(\mathcal{T}) = \{ \mathrm{in}_{\mathbf{w}}(f) : f \in \mathcal{T} \}$ contains a unit.

With a tropical basis, every tropical variety is a tropical prevariety.

Corollary (Corollary 3.2.3)

Every tropical variety is a finite intersection of tropical hypersurfaces.

More precisely, if $\mathcal{T}$ is a tropical basis of the ideal I, then

$$\mathrm{trop}(X) = \bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f)).$$

Corollary (Corollary 3.2.4)

If X is a subvariety of the torus T^n over K, then its tropicalization $\mathrm{trop}(X)$ is the support of a $\Gamma_{\rm val}$-rational polyhedral complex.

the fundamental theorem

Theorem (Fundamental Theorem of Tropical Algebraic Geometry)

Let I be an ideal in $K[\mathbf{x}^{\pm 1}]$ and let $X = V(I)$ its variety in the algebraic torus $T^n \cong (K^*)^n$.

Then the following three subsets of $\mathbb{R}^n$ coincide:

1. *the tropical variety* $\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f))$;
2. *the closure in $\mathbb{R}^n$ of the set of all vectors $\mathbf{w} \in \Gamma_{\mathrm{val}}^n$ with $\mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle$;*
3. *the closure of the set of coordinatewise valuations of points in X:*

$$\mathrm{val}(X) = \{ (\mathrm{val}(z_1), \mathrm{val}(z_2), \ldots, \mathrm{val}(z_n)) : (z_1, z_2, \ldots, z_n) \in X \}.$$

initial forms and monomial maps

Lemma (Lemma 3.2.6)

Let $X \subset T^n$ be an irreducible variety of dimension d, with prime ideal $I \subset K[\mathbf{x}^{\pm 1}]$ and let $\mathbf{w} \in \mathrm{trop}(X) \cap \Gamma^n_{\mathrm{val}}$. All minimal associated primes of $\mathrm{in}_{\mathbf{w}}(I)$ in $\mathbb{K}[\mathbf{x}^{\pm 1}]$ have dimension d.

Proposition (Proposition 3.2.7)

Let X be a subvariety in T^n and $m \geq \dim(X)$. There is a monomial map $\phi : T^n \rightarrow T^m$ with its image $\phi(X)$ Zariski closed in T^m and $\dim(\phi(X)) = \dim(X)$. We can choose this map so that the kernel of the induced linear map $\mathrm{trop}(\phi) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ intersects trivially with a fixed finite arrangement of codimension $n - m$ subspaces in $\mathbb{R}^n$.

The proof derives a version of Noether normalization for $K[\mathbf{x}^{\pm 1}]$.

the Gröbner characterization

Proposition (Proposition 3.2.8)

Let I be an ideal in $K[\mathbf{x}^{\pm 1}]$ and $X = V(I)$ its variety.

Then $\mathrm{trop}(X)$ is the union of all cells in the Gröbner complex $\Sigma(I_{\mathrm{proj}})$.

Lemma (Lemma 3.2.10)

Let X be a d-dimensional subvariety of T^n, with ideal $I \subset K[\mathbf{x}^{\pm 1}]$.

Every polyhedron in the Gröbner complex with support $\{ \mathbf{w} \in \Gamma_{\mathrm{val}}^n : \mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle \}$ has dimension at most d.

lifting points and monomial maps

Proposition (Proposition 3.2.11)

Let X be an irreducible d-dimensional subvariety of T^n with prime ideal $I \subseteq K[\mathbf{x}^{\pm 1}]$. Fix $\mathbf{w} \in \Gamma_{\rm val}^n$ with $\mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle$ and $\mathbf{z} \in V(\mathrm{in}_{\mathbf{w}}(I)) \subset (\mathbb{K}^*)^n$.

There is a $\mathbf{y} \in X$ with $\mathrm{val}(\mathbf{y}) = \mathbf{w}$ and $\overline{t^{-\mathbf{w}}\mathbf{y}} = \mathbf{z}$.

If $\dim(X) > 0$, then there are infinitely many such $\mathbf{y} \in X$.

Tropicalization commutes with morphism of tori:

Corollary (Corollary 3.2.13)

Let $\phi : T^n \rightarrow T^m$ be a monomial map. Consider any subvariety X of T^n and the Zariski closure $\overline{\phi(X)}$ of its image in T^m. Then:

$$\mathrm{trop}(\overline{\phi(X)}) = \mathrm{trop}(\phi)(\mathrm{trop}(\mathrm{X})).$$

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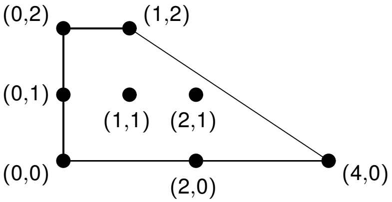
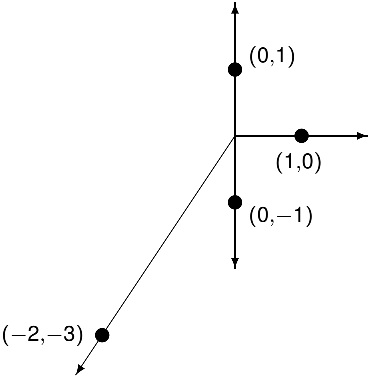
an example

Consider $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, a Laurent polynomial:

$$f = c_{1,2}xy^2 + c_{0,2}y^2 + c_{2,1}x^2y + c_{1,1}xy + c_{1,0}y + c_{4,0}x^4 + c_{2,0}x^2 + c_{0,0}$$

with its Newton polygon and its normal fan:





multiplicity

Definition

Let $S = \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. The primary decomposition of an ideal I in S is a finite intersection of primary ideals: $I = \bigcap_{i=1}^{r} Q_i$ with corresponding irreducible decomposition $I = \bigcap_{j=1}^{s} P_j$, with P_j the minimal associated primes, obtained as $P_j = \sqrt{Q_i}$ for some i, $P_j \in \mathrm{Ass}(I)$.

The *multiplicity of P_j* is

$$\mathrm{mult}(P_j, I) := \ell((S/Q_i)_{P_j}) = \ell((I : P_j^\infty)/I)_{P_j}),$$

where $\ell(M)$ is the length of an S-module M.

weights on a fan

Definition (Definition 3.4.3)

Let I be an ideal in $K[x_1^{\pm 1}, x_2^1, \ldots, x_n^{\pm 1}]$.

Let Σ be a polyhedral complex with support $|\Sigma| = \mathrm{trop}(V(I))$ such that $\mathrm{in}_{\mathbf{w}}(I)$ is constant for $\mathbf{w} \in \mathrm{relint}(\sigma)$ for all $\sigma \in \Sigma$.

For a $\sigma \in \Sigma$, maximal with respect to inclusion, its *multiplicity* is

$$\mathrm{mult}(\sigma) = \sum_{P \in \mathrm{Ass}(I)} \mathrm{mult}(P, \mathrm{in}_{\mathbf{w}}(I)) \quad \text{for any } \mathbf{w} \in \mathrm{relint}(\sigma).$$

a balanced fan

Definition

Let Σ be a rational, pure d-dimensional fan in $\mathbb{R}^n$.
Fix weights $m(\sigma) \in \mathbb{N}^n$ for all d-dimensional cones $\sigma \in \Sigma$.

For a $(d-1)$-dimensional cone $\tau \in \Sigma$,
let L be the linear space parallel to τ, $\dim(L) = d - 1$. The abelian group $L_\mathbb{Z} = L \cap \mathbb{Z}^n$ is free of rank $d-1$, with $N_\tau = \mathbb{Z}^n / L_\mathbb{Z} \cong \mathbb{Z}^{n-d+1}$.

For each $\sigma \in \Sigma$ with $\tau \subsetneq \sigma$, the set $(\sigma + L)/L$ is a one dimensional cone in $N_r \otimes \mathbb{R}$. Let $\mathbf{u}_\sigma$ be the first lattice point on this ray.

The fan Σ is *balanced at* τ if $\displaystyle\sum_{\sigma \supsetneq \tau} m(\sigma)\mathbf{u}_\sigma = 0$.

The fan Σ is *balanced* if it is balanced at each $\tau \in \Sigma$, $\dim(\tau) = d - 1$.

the structure theorem

Definition

A pure d-dimensional polyhedral complex Σ in $\mathbb{R}^n$ *is connected through codimension one* if for any two d-dimensional cells $P, Q \in \Sigma$ there is a chain $P = P_1, P_2, \ldots, P_s = Q$ for which P_i and P_{i+1} share a common facet F_i, for $1 \leq i < s$. Since P_i are facets of Σ and F_i are ridges, we call this a *facet-ridge path* connecting P and Q.

Theorem (Structure Theorem for Tropical Varieties)

Let X be an irreducible d-dimensional variety of T^n.
Then $\mathrm{trop}(X)$ is the support of a balanced weighted Γ_{val}-rational polyhedral complex pure of dimension d.
Moreover, the polyhedral complex is connected through codimension 1.

computing multiplicities

Lemma (Lemma 3.4.6)

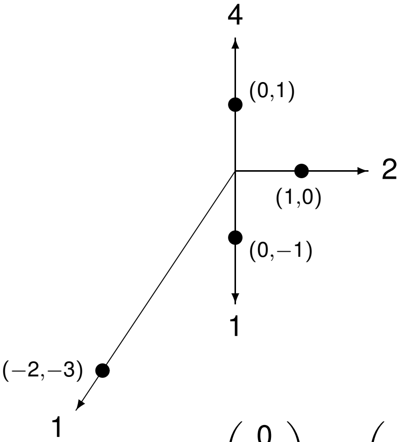
Let $f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ *and*

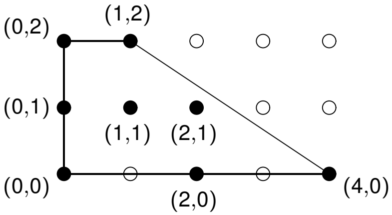
- *let* Δ *be a regular subdivision of the Newton polytope of* f*, induced by* $\mathrm{val}(c_{\mathbf{a}})$*; and*
- *let* Σ *be the polyhedral complex supported on* $\mathrm{trop}(V(f))$ *that is dual to* Δ*.*

The multiplicity of a maximal cell $\sigma \in \Sigma$ *is the lattice length of the edge* $e(\sigma)$ *of* Δ *dual to* σ*.*

the example revisited

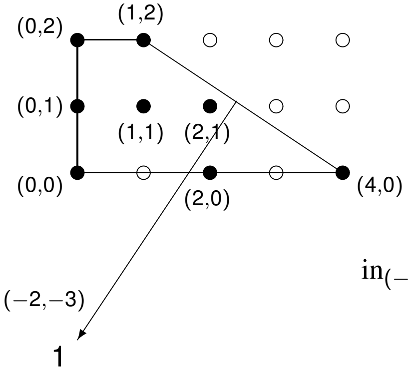
Assume $\mathrm{val}(\cdot)$ does not triangulate the Newton polygon of f.





$$4\begin{pmatrix}0\\1\end{pmatrix}+2\begin{pmatrix}1\\0\end{pmatrix}+1\begin{pmatrix}0\\-1\end{pmatrix}+1\begin{pmatrix}-2\\-3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

applying a unimodular coordinate transformation



$$\mathrm{in}_{(-2,-3)}(f)(x,y) = x^4 + xy^2$$

$$U = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}, \det(U) = 1$$

$$U \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -8 & -8 \\ 4 & 3 \end{pmatrix}$$

$$\begin{aligned} \mathrm{in}_{(-2,-3)}(f)(x = X^{-2}Y^1, y = X^{-3}Y^1) \\ = (X^{-2}Y^1)^4 + (X^{-2}Y^1)(X^{-3}Y^1)^2 \\ = X^{-8}Y^4 + X^{-8}Y^3 \\ = X^{-8}Y^3(Y+1) \end{aligned}$$

proof of Lemma 3.4.6

- Pick $\mathbf{w}$ in the relative interior of σ. The initial ideal $\mathrm{in}_{\mathbf{w}}(\langle f \rangle)$ is generated by
$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\mathbf{a} \in e(\sigma)} \overline{t^{-\mathrm{val}(c_{\mathbf{a}})}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}.$$
- Since $\dim(e(\sigma)) = 1$, $\mathbf{a} - \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in e(\sigma)$ is unique up to scaling, so take $\mathbf{v} = \mathbf{a} - \mathbf{b}$ of minimal length.
- $\mathrm{in}_{\mathbf{w}}(f)$ is then a monomial times $g \in K[\mathbf{x}^{\pm 1}]$ in the variable $y = \mathbf{x}^{\mathbf{v}}$.
- We may multiply f with a monomial so $\mathrm{in}_{\mathbf{w}}(f)$ is a polynomial (without negative exponents) with nonzero constant term.
- $\deg(g)$ equals the lattice length of $e(\sigma)$, which equals the multiplicity of σ. □

balancing with multiplicities

Theorem (Theorem 3.4.14)

Let I be an ideal in $K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that all irreducible components of $V(I)$ have the same dimension d.

Fix a polyhedral complex Σ with support $\mathrm{trop}(V(I))$ such that $\mathrm{in}_{\mathbf{w}}(I)$ is constant for $\mathbf{w}$ in the relative interior of each cell in Σ.

Then Σ is a weighted balanced polyhedral complex with the weight function `mult` of Definition 3.4.3.