Tropical Islands

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Graduate Computational Algebraic Geometry Seminar

Tropical Islands

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Introduction to Tropical Geometry

Introduction to Tropical Geometry is the title of a forthcoming book of Diane Maclagan and Bernd Sturmfels.

The web page
`http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.html`
offers the pdf file of the first five chapters (23 August 2013).

Tropical islands is the title of the first chapter,
which promises a friendly welcome to tropical mathematics.

overview of the book

The titles of the five chapters with some important sections:

1. Tropical Islands
   - amoebas and their tentacles
   - implicitization
2. Building Blocks
   - polyhedral geometry
   - Gröbner bases
   - tropical bases
3. Tropical Varieties
   - the fundamental theorem
   - the structure theorem
   - multiplicities and balancing
   - connectivity and fans
   - stable intersection
4. Tropical Rain Forest
5. Linear Algebra

Tropical Islands

the min-plus algebra

In the min-plus algebra, addition is replaced by the minimum and the multiplication is replaced by the addition.

Imre Simon pioneered the min-plus algebra in optimization theory. French mathematicians called the min-plus algebra tropical.

Origins of algebraic geometry: study zero sets of polynomial systems.

Polynomials in the tropical semiring define piecewise-linear functions. Tropical algebraic varieties are composed of convex polyhedra.

Tropical methods are used to solve problems in algebra, geometry, and combinatorics.

the tropical semiring

The tropical semiring is denoted as $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, with

$$x \oplus y = \min(x, y) \quad \text{and} \quad x \odot y = x + y.$$

The neutral elements are ∞ and 0: $x \oplus \infty = x$ and $x \odot 0 = x$.

Both operations are commutative and associative.
The distributive law is $x \odot (y \oplus z) = x \odot y \oplus y \odot z$.

There is no substraction, e.g.: $4 \oplus x = 13$ has no solution.

the Freshman's Dream

The tropical Pascal's triangle with binomial coefficients is zero.

$$
\begin{aligned}
(x \oplus y)^{\odot 2} & = (x \oplus y) \odot (x \oplus y) \\
& = \min(x, y) + \min(x, y) \\
& = \min(x + x, x + y, y + x, y + y) \\
& = \min(x + x, y + y) \\
& = \min(x^{\odot 2}, y^{\odot 2}) \\
& = x^{\odot 2} \oplus y^{\odot 2}
\end{aligned}
$$

For any power p: $(x \oplus y)^{\odot p} = x^{\odot p} \oplus y^{\odot p}$.

tropical monomials

Let $x_1, x_2, \dots, x_n$ be variables representing elements in $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, then a tropical monomial is a product of variables, allowing repetition. Example:

$$x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_4 \odot x_2 \odot x_3 \odot x_2 = x_1^{\odot 2} x_2^{\odot 3} x_3^{\odot 2} x_4.$$

As a function, a tropical monomial is a linear function.

$$x_2 + x_1 + x_3 + x_1 + x_4 + x_2 + x_3 + x_2 = 2x_1 + 3x_2 + 2x_3 + x_4.$$

Every linear function in n variables with integer coefficients we can write as a tropical monomial, exponents can be negative.

Tropical monomials are the linear functions on $\mathbb{R}^n$ with integer coefficients.

tropical polynomials

Any *finite* linear combination of tropical monomials defines a tropical polynomial with real coefficients and integer exponents:

$$\begin{aligned} p(x_1, x_2, \ldots, x_n) & = & c_{\mathbf{a}_1} \odot x_1^{\odot a_{1,1}} x_2^{\odot a_{1,2}} \cdots x_n^{\odot a_{1,n}} \\ & \oplus & c_{\mathbf{a}_2} \odot x_1^{\odot a_{2,1}} x_2^{\odot a_{2,2}} \cdots x_n^{\odot a_{2,n}} \\ & \oplus & \cdots \\ & \oplus & c_{\mathbf{a}_k} \odot x_1^{\odot a_{k,1}} x_2^{\odot a_{k,2}} \cdots x_n^{\odot a_{k,n}}. \end{aligned}$$

The corresponding function:

$$\begin{aligned} p(x_1, x_2, \ldots, x_n) = \min(& c_{\mathbf{a}_1} + a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n, \\ & c_{\mathbf{a}_2} + a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n, \\ & \ldots, \\ & c_{\mathbf{a}_k} + a_{k,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,n}x_n \quad). \end{aligned}$$

piecewise-linear functions

Tropical polynomials as functions $p : \mathbb{R}^n \rightarrow \mathbb{R}$

1. are continuous;
2. are piecewise-linear, with a finite number of pieces; and
3. are concave, i.e.: $p\left(\frac{x+y}{2}\right) \geq \frac{1}{2}(p(x) + p(y))$ for all $x, y \in \mathbb{R}$.

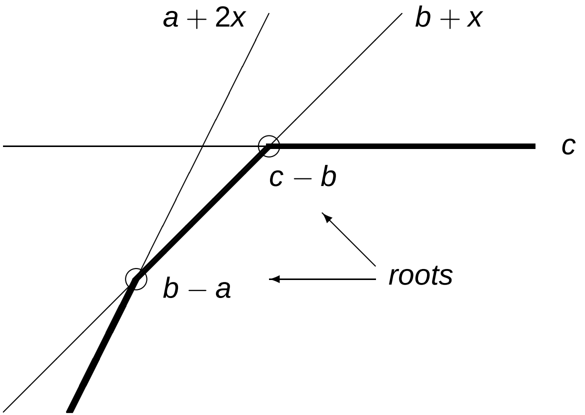
Any function which satisfies these three properties can be represented as the minimum of a finite set of linear functions.

Lemma

The tropical polynomials in n variables are the piecewise-linear functions on $\mathbb{R}^n$ with integer coefficients.

the graph of a tropical quadratic polynomial

$$a \odot x^{\odot 2} \oplus b \odot x \oplus c = \min(a + 2x, b + x, c)$$



If $b - a \leq c - b$, then
$$\begin{aligned} p(x) &= a \odot (x \oplus (b - a)) \odot (x \oplus (c - b)) \\ &= a + \min(x, b - a) + \min(x, c - b). \end{aligned}$$

the tropical fundamental theorem of algebra

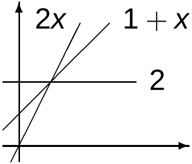
Theorem

Every tropical polynomial <u>function</u> can be written uniquely as a tropical product of tropical linear functions.

Note:

- Distinct polynomials can represent the same function.

$$
\begin{aligned}
x^2 \oplus 17 \odot x \oplus 2 &= \min(2x, x+17, 2) \\
&= \min(2x, x+1, 2) \\
&= x^2 \oplus 1 \odot x \oplus 2 \\
&= (x \oplus 1)^{\odot 2}
\end{aligned}
$$



- Every polynomial can be replaced by an equivalent polynomial (equivalent means: representing the same function) that can be factored into linear functions.

Tropical Islands

dynamic programming

Problem: find the shortest path in a weighted directed graph.

Graph G with n nodes labeled $1, 2, \ldots, n$;
edge (i, j) has weight $d_{ij} \geq 0$; $d_{ii} = 0$ for all nodes i;
if no edge between i and j, then $d_{ij} = \infty$.

$D_G = (d_{ij})$ is the adjacency matrix of G.

The Floyd-Warshall algorithm uses the recursive formula
for finding the shortest path, for $r \geq 2$:

$$d_{ij}^{(r)} = \min\{ d_{ik}^{(r-1)} + d_{kj} : k = 1, 2, \ldots, n \}$$

is the length of the shortest path between i and j,
visiting at most r edges, where $d_{ij}^{(1)} = d_{ij}$.

tropical formulation

The formula

$$d_{ij}^{(r)} = \min\{ \, d_{ik}^{(r-1)} + d_{kj} : k = 1, 2, \ldots, n \, \}$$

can be rewritten, using $\min = \oplus$ and $+ = \odot$, as follows:

$$\begin{aligned} d_{ij}^{(r)} &= d_{i1}^{(r-1)} \odot d_{1j} \oplus d_{i2}^{(r-1)} \odot d_{2j} \oplus \cdots \oplus d_{in}^{(r-1)} \odot d_{nj} \\ &= (d_{i1}^{(r-1)}, d_{i2}^{(r-1)}, \ldots, d_{in}^{(r-1)}) \odot (d_{1j}, d_{2j}, \ldots, d_{nj})^T. \end{aligned}$$

The right hand side of the formula for $d_{ij}^{(r)}$ is the product of the ith row of $D_G^{\odot r-1}$ with the jth column of D_G.

Proposition

The entry of $D_G^{\odot n-1}$ in row i and column j equals the length of the shortest path from i to j.

from classical to tropical arithmetic

To D_G associate $A_G(\epsilon)$, for an infinitesimal $\epsilon > 0$, e.g.:

$$D_G = \begin{pmatrix} 0 & 1 & 3 & 7 \\ 2 & 0 & 1 & 3 \\ 4 & 5 & 0 & 1 \\ 6 & 3 & 1 & 0 \end{pmatrix} \quad A_G(\epsilon) = \begin{pmatrix} 1 & \epsilon^1 & \epsilon^3 & \epsilon^7 \\ \epsilon^2 & 1 & \epsilon^1 & \epsilon^3 \\ \epsilon^4 & \epsilon^5 & 1 & \epsilon^1 \\ \epsilon^6 & \epsilon^3 & \epsilon^1 & 1 \end{pmatrix}$$

Then $A_G(\epsilon)^3$ is in classical arithmetic:

$$\begin{pmatrix} 1+3\epsilon^3+\cdots & 3\epsilon+\epsilon^4+\cdots & 3\epsilon^2+3\epsilon^3+\cdots & \epsilon^3+6\epsilon^4+\cdots \\ 3\epsilon^2+4\epsilon^5+\cdots & 1+3\epsilon^3+\cdots & 3\epsilon+\epsilon^3+\cdots & 3\epsilon^2+3\epsilon^3+\cdots \\ 3\epsilon^4+2\epsilon^6+\cdots & 3\epsilon^4+6\epsilon^5+\cdots & 1+3\epsilon^2+\cdots & 3\epsilon+\epsilon^3+\cdots \\ 6\epsilon^5+3\epsilon^6+\cdots & 3\epsilon^3+\epsilon^5+\cdots & 3\epsilon+\epsilon^3+\cdots & 1+3\epsilon^2+\cdots \end{pmatrix}$$

The lowest exponent of ϵ in the (i,j)-th entry of $A_G(\epsilon)^3$ is the (i,j)-th entry in the tropical matrix $D_G^{\odot 3}$.

tropicalization

Passing from classical to tropical arithmetic is *tropicalization*, informally summarized as

$$\text{tropical} = \lim_{\epsilon \longrightarrow 0} \log_\epsilon(\text{classical}(\epsilon)).$$

The algebraic notion of *valuations* make tropicalization rigorous.

the assignment problem

Imagine n jobs have to be assigned to n workers; and
each job needs to be assigned to exactly one worker.

Notations:

- Let x_{ij} be the cost of assigning job i to worker j.
- An assignment is a permutation π of $(1, 2, \ldots, n)$.
- S_n is the group of all permutations of n elements.

The optimal cost is the minimum over all permutations:

$$\min\{\ x_{1\pi(1)} + x_{2\pi(2)} + \cdots + x_{n\pi(n)} : \pi \in S_n\ \}.$$

The *tropical permanent* of a matrix $X = (x_{ij})$ is

$$\mathrm{tropdet}(X) = \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)}.$$

the tropical determinant = the tropical permanent

$$\text{tropdet}(X) = \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)}$$

As the sum over the products defined by all permutations, the tropical permanent is the same as the tropical permanent.

Proposition

The tropical determinant solves the assignment problem.

The Hungarian assignment method (Harold Kuhn, 1955) is

- an iterative method that at each step chooses an unassigned worker and a shortest path from this worker to the set of jobs;
- is a polynomial-time algorithm, runs in $O(n^3)$ operations;
- is a certain tropicalization of Gaussian elimination.

tropicalization

Consider a matrix in ϵ, with terms of lowest order listed first:

$$A(\epsilon) = \left(\begin{array}{ccc} a_{11}\epsilon^{x_{11}} + \cdots & a_{12}\epsilon^{x_{12}} + \cdots & a_{13}\epsilon^{x_{13}} + \cdots \\ a_{21}\epsilon^{x_{21}} + \cdots & a_{22}\epsilon^{x_{22}} + \cdots & a_{23}\epsilon^{x_{23}} + \cdots \\ a_{31}\epsilon^{x_{31}} + \cdots & a_{32}\epsilon^{x_{32}} + \cdots & a_{33}\epsilon^{x_{33}} + \cdots \end{array} \right).$$

For sufficiently random values of a_{ij}, so no cancellation of lowest order coefficients happens when expanding $\det(A(\epsilon))$, we have

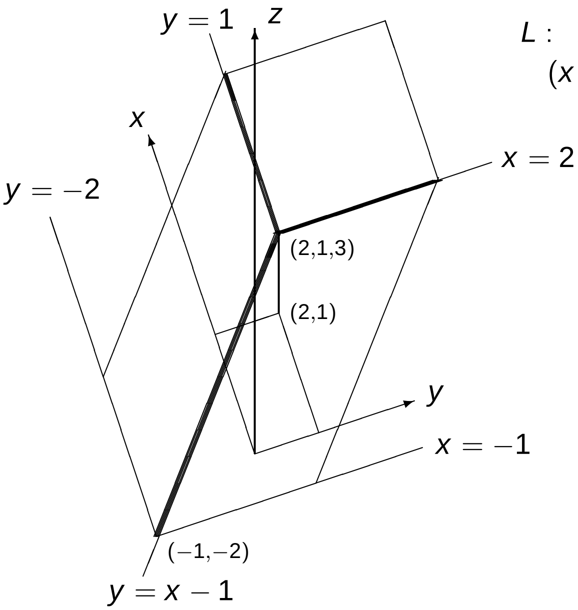
$$\det(A(\epsilon)) = \alpha \cdot \epsilon^{\mathrm{tropdet}(X)} + \cdots \quad \text{for some nonzero } \alpha.$$

To compute $\mathrm{tropdet}(X)$ we can apply tropicalization:

$$\text{tropical} = \lim_{\epsilon \rightarrow 0} \log_\epsilon(\text{classical}(\epsilon)).$$

Tropical Islands

graph of the tropical line $L(x,y) = 1 \odot x \oplus 2 \odot y \oplus 3$



$$L : \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$(x, y) \mapsto \min(1 + x, 2 + y, 3)$$

$$\left.\begin{array}{l} z = 1 + x \\ z = 2 + y \end{array}\right\} : y = x - 1$$

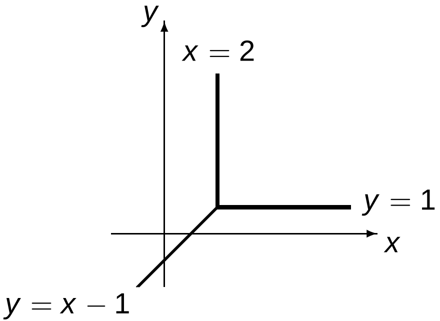
$$\left.\begin{array}{l} z = 1 + x \\ z = 3 \end{array}\right\} : x = 2$$

$$\left.\begin{array}{l} z = 2 + y \\ z = 3 \end{array}\right\} : y = 1$$

a picture in the plane

$$L : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \min(1 + x, 2 + y, 3)$$

Look where the minimum is attained at least twice:



intersecting two tropical lines

Intersecting $1 \odot x \oplus 2 \odot y \oplus 3$ with $-2 \odot x \oplus 4 \odot y \oplus 2$: where $\min(1+x, 2+y, 3)$ and $\min(-2+x, 4+y, 2)$ attain their mininum twice.