Initial Forms and Gröbner Polyhedra

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Graduate Computational Algebraic Geometry Seminar

Gröbner Complexes and Tropical Bases

Initial Forms and Gröbner Polyhedra

Introduction to Tropical Geometry

Introduction to Tropical Geometry is the title of a forthcoming book of Diane Maclagan and Bernd Sturmfels.

The web page
`http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.html`
offers the pdf file of a book, dated 31 March 2014.

Today we look at some building blocks ...

This seminar is based on sections 2.4 and 2.5.

Initial Forms and Gröbner Polyhedra

Gröbner Bases over Fields with Valuations

The *initial ideal* of a homogeneous ideal I in $K[x_0, x_1, \ldots, x_n]$ is $\mathrm{in}_{\mathbf{w}}(I) = \langle\ \mathrm{in}_{\mathbf{w}}(f)\ :\ f \in I\ \rangle \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$, $\mathbb{K}$ is the residue field.

A *Gröbner basis* for I with respect to $\mathbf{w}$ is

- a finite set $\mathcal{G} = \{\ g_1, g_2, \ldots, g_s\ \} \subset I$,
- with $\langle\ \mathrm{in}_{\mathbf{w}}(g_1), \mathrm{in}_{\mathbf{w}}(g_2), \ldots, \mathrm{in}_{\mathbf{w}}(g_s)\ \rangle = \mathrm{in}_{\mathbf{w}}(I)$.

Lemma (Lemma 2.4.2)

Let $I \subset K[x_0, x_1, \ldots, x_n]$ be a homogeneous ideal and fix $\mathbf{w} \in (\Gamma_{\mathrm{val}})^{n+1}$. Then $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous and we may choose a homogeneous Gröbner basis for I.
Furthermore, if $g \in \mathrm{in}_{\mathbf{w}}(I)$, then $g = \mathrm{in}_{\mathbf{w}}(f)$ for some $f \in I$.

initial forms of initial forms of polynomials

The initial form of an initial form is an initial form.

Lemma (Lemma 2.4.5)

Fix $f \in K[x_0, x_1, \ldots, x_n]$, $\mathbf{w} \in \Gamma_{\rm val}^{n+1}$, and $\mathbf{v} \in \mathbb{Q}^{n+1}$.
There exists an $\epsilon > 0$ such that for all $\delta \in \Gamma_{\rm val}$ with $0 < \delta < \epsilon$, we have

$$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f)) = \mathrm{in}_{\mathbf{w}+\delta\mathbf{v}}(f).$$

Lemma (Lemma 2.4.6)

Let I be a homogeneous ideal in $K[x_0, x_1, \ldots, x_n]$ and fix $\mathbf{w} \in \Gamma_{\rm val}^{n+1}$.
There exists a $\mathbf{v} \in \mathbb{Q}^{n+1}$ and $\epsilon > 0$ such that

1. $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ *and* $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$ *are monomial ideals; and*
2. $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$.

$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is a monomial ideal

Proof that $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is a monomial ideal:

- Denote by $M_{\mathbf{v}}$ the monomial ideal $\langle \mathbf{x}^{\mathbf{a}} : \mathbf{x}^{\mathbf{a}} \in \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))\rangle$, with $\mathbf{v}$ chosen such that $M_{\mathbf{v}}$ is maximal, polynomial rings are Noetherian.
- Suppose $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is not a monomial ideal. Then there is a $f \in I$ such that none of the terms of $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f))$ lies in $M_{\mathbf{v}}$.
- For generic $\mathbf{u} \in \mathbb{Q}^{n+1}$, $\mathrm{in}_{\mathbf{u}}(\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f)))$ is a monomial, with its exponents corresponding to a vertex of the Newton polytope of f.

  By Lemma 2.4.5, for some $\epsilon > 0$, for all $0 < \delta < \epsilon$:
  $$\mathrm{in}_{\mathbf{u}}(\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f))) = \mathrm{in}_{\mathbf{v}+\delta\mathbf{u}}(\mathrm{in}_{\mathbf{w}}(f)).$$
- For sufficiently small δ, $\mathrm{in}_{\mathbf{v}+\delta\mathbf{u}}(I)$ contains each generator of $\langle \mathbf{x}^{\mathbf{a}} : \mathbf{x}^{\mathbf{a}} \in \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))\rangle$, as $\mathbf{x}^{\mathbf{a}} = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f))$ for some $f \in I$ (this follows from Lemma 2.4.5). But, by choice of $\mathbf{v}$, $M_{\mathbf{v}}$ is maximal.

By this contradition, $M_{\mathbf{v}} = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$.

$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$

Proof that $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$:

- Now we can write $M_{\mathbf{v}} = \langle \mathbf{x}^{\mathbf{a}_1}, \mathbf{x}^{\mathbf{a}_2}, \ldots, \mathbf{x}^{\mathbf{a}_s} \rangle$, with chosen f_i: $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f_i)) = \mathbf{x}^{\mathbf{a}_i}$, for $i = 1, 2, \ldots, s$.
- By Lemma 2.4.5, there is an $\epsilon > 0$: $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(f_i) = \mathbf{x}^{\mathbf{a}_i}$ for all i. Therefore, for this ϵ we have $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$.
- Choose $\mathbf{v} \in \mathbb{Q}^{n+1}$:
  1. $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is a monomial ideal; and
  2. $M^{\epsilon}_{\mathbf{v}} = \langle \mathbf{x}^{\mathbf{a}} : \mathbf{x}^{\mathbf{a}} \in \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) \rangle$ is maximal.
- Suppose $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$ is not a monomial ideal. Then there is an $f \in I$ with no term of $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(f)$ in $M^{\epsilon}_{\mathbf{v}}$. As before we choose a $\mathbf{u}$ so that $M^{\epsilon}_{\mathbf{v}} \subsetneq M^{\epsilon}_{\mathbf{v}+\delta\mathbf{u}}$ for small $\delta > 0$. For small $\delta > 0$: $M^{\epsilon}_{\mathbf{v}+\delta\mathbf{u}} = \langle \mathbf{x}^{\mathbf{a}} : \mathbf{x}^{\mathbf{a}} \in \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) \rangle$, a contradiction.

Thus, $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$ is a monomial ideal and then $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$.

Hilbert functions and dimension

$S_K = K[x_0, x_1, \ldots, x_n]$ and $S_{\mathbb{K}} = \mathbb{K}[x_0, x_1, \ldots, x_n]$ contain homogeneous ideals I and their initial forms $\mathrm{in}_{\mathbf{w}}(I)$.

The Hilbert function $\mathbb{N} \rightarrow \mathbb{N} : d \mapsto \dim(S_K/I)_d$
maps the degree d to the dimension of the quotient of the ring S_K modulo I, restricted to polynomials of degree d.

For large enough d, the Hilbert function is a polynomial in d.

Lemma (Lemma 2.4.7)

Let I be a homogeneous ideal in S_K and let $\mathbf{w} \in \Gamma_{\mathrm{val}}^{n+1}$ be such that $\mathrm{in}_{\mathbf{w}}(I)_d$ is spanned over K by its monomials.
The monomials of degree d that are not in $\mathrm{in}_{\mathbf{w}}(I)$ form a K-basis for $(S/I)_d$.

linear independence

Let $\mathcal{B}_d$ be the set of monomials of degree d not in $\mathrm{in}_{\mathbf{w}}(I)$.

Suppose $\mathcal{B}_d$ is linearly dependent over K:

- There is a $f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} x^{\mathbf{a}} \in I_d$ with $\mathbf{x}^{\mathbf{a}} \not\in \mathrm{in}_{\mathbf{w}}(I)$, as $\mathbf{x}^{\mathbf{a}} \in \mathcal{B}_d$.
- However, $\mathrm{in}_{\mathbf{w}}(f) \in \mathrm{in}_{\mathbf{w}}(I)_d$, every term of $\mathrm{in}_{\mathbf{w}}(f)$ is in $\mathrm{in}_{\mathbf{w}}(I)_d$ which contradicts the construction of f.

The linear independence of $\mathcal{B}_d$ implies

$$\dim_{\mathbb{K}} \mathrm{in}_{\mathbf{w}}(I)_d \geq \dim_K (I)_d \quad \text{because} \quad |\mathcal{B}_d| = \left(\begin{array}{c} n+d \\ n \end{array} \right) - \dim \mathrm{in}_{\mathbf{w}}(I)_d.$$

$\mathcal{B}_d$ forms a K-basis

By Lemma 2.4.2, for each monomial $\mathbf{x}^\mathbf{a} \in \mathrm{in}_\mathbf{w}(I)_d$, we can choose a $f_\mathbf{a} \in I_d$ with $\mathrm{in}_\mathbf{w}(f_\mathbf{a}) = \mathbf{x}^\mathbf{a}$. Consider $\{ f_\mathbf{a} : \mathbf{x}^\mathbf{a} \in \mathrm{in}_\mathbf{w}(I)_d \}$.

Suppose $\{ f_\mathbf{a} : \mathbf{x}^\mathbf{a} \in \mathrm{in}_\mathbf{w}(I)_d \}$ is not linearly independent in S_K/I:

- There are $\gamma_\mathbf{a} \in K^*$: $\displaystyle\sum_{\mathbf{a} \in A} \gamma_\mathbf{a} f_\mathbf{a} = 0$.
- Write $f_\mathbf{a} = \mathbf{x}^\mathbf{a} + \displaystyle\sum_\mathbf{b} c_{\mathbf{ab}} \mathbf{x}^\mathbf{b}$ and let $\mathbf{u}$ be where $\mathrm{val}(\gamma_\mathbf{a}) + \langle \mathbf{w}, \mathbf{a} \rangle$ is minimal for all $\mathbf{a} \in A$ with $\mathbf{x}^\mathbf{a} \in \mathrm{in}_\mathbf{w}(I)_d$.
- Then $\gamma_\mathbf{u} + \displaystyle\sum_{\mathbf{b} \neq \mathbf{u}} \gamma_\mathbf{b} c_{\mathbf{bu}} = 0$, so there is a $\mathbf{v} \neq \mathbf{u}$ with
$$\mathrm{val}(\gamma_\mathbf{v}) + \mathrm{val}(c_{\mathbf{vu}}) \leq \mathrm{val}(\gamma_\mathbf{u}).$$
- But then $\mathrm{val}(\gamma_\mathbf{v}) + \mathrm{val}(c_{\mathbf{vu}}) + \langle \mathbf{w}, \mathbf{u} \rangle \leq \mathrm{val}(\gamma_\mathbf{u}) + \langle \mathbf{w}, \mathbf{u} \rangle$ and $\mathrm{val}(\gamma_\mathbf{u}) + \langle \mathbf{w}, \mathbf{u} \rangle \leq \mathrm{val}(\gamma_\mathbf{v}) + \langle \mathbf{w}, \mathbf{v} \rangle$, which contradicts $\mathrm{in}_\mathbf{w}(f_\mathbf{v}) = \mathbf{x}^\mathbf{v}$.

This shows $\dim_K I_d \geq \dim_\mathbb{K} \mathrm{in}_\mathbf{w}(I)_d$. Thus $\dim_K (S_K/I)_d = \dim_\mathbb{K}(S_\mathbb{K}/\mathrm{in}_\mathbf{w}(I))_d$, and $\mathcal{B}_d$ is a K-basis for $(S_K/I)_d$.

two corollaries

Corollary (Corollary 2.4.8)

For any $\mathbf{w} \in \Gamma_{\mathrm{val}}^{n+1}$ *and any homogeneous ideal* I *in* S_K*, the Hilbert function of* I *agrees with that of its initial ideal* $\mathrm{in}_{\mathbf{w}}(I)$ *in* $S_{\mathbb{K}}$*, i.e.:*

$$\dim_K(S_K/I)_d = \dim_{\mathbb{K}}(S_{\mathbb{K}}/\mathrm{in}_{\mathbf{w}}(I))_d \quad \textit{for all } d \geq 0.$$

Thus the Krull dimensions of the rings S_K/I *and* $S_{\mathbb{K}}/\mathrm{in}_{\mathbf{w}}(I)$ *coincide.*

The Krull dimension of a ring is the supremum of the lengths of chains of distinct prime ideals in the ring.

Corollary (Corollary 2.4.9)

Let I *be a homogeneous ideal in* $K[x_0, x_1, \ldots, x_n]$.
For any $\mathbf{w} \in \Gamma_{\mathrm{val}}^{n+1}$ *and* $\mathbf{v} \in \mathbb{Q}^{n+1}$ *there exists* $\epsilon > 0$ *such that*

$$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}+\delta\mathbf{v}}(I) \quad \textit{for all } 0 < \delta < \epsilon \textit{ with } \delta\mathbf{v} \in \Gamma_{\mathrm{val}}^{n+1}.$$

Initial Forms and Gröbner Polyhedra

defining polyhedra

For a homogeneous ideal $I \subset K[x_0, x_1, \ldots, x_n]$ and for $\mathbf{w} \in \Gamma_{\rm val}^{n+1}$ we set

$$C_I[\mathbf{w}] = \{ \, \mathbf{v} \in \Gamma_{\rm val}^{n+1} : {\rm in}_{\mathbf{v}}(I) = {\rm in}_{\mathbf{w}}(I) \, \}.$$

Let $\overline{C_I[\mathbf{w}]}$ be the closure of $C_I[\mathbf{w}]$ in $\mathbb{R}^{n+1}$ in the Euclidean topology.

Consider a Gröbner basis $\{g_1, g_2, \ldots, g_s\}$ of I with respect to $\mathbf{w}$, and let ${\rm in}_{\mathbf{w}}(g_i) = \mathbf{x}^{\mathbf{u}_i}$, for $g_i = \displaystyle\sum_{\mathbf{a} \in \mathbb{N}^{n+1}} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}$.

If $\overline{C_I[\mathbf{w}]}$ has the inequality description

$$\{ \, \mathbf{z} \in \mathbb{R}^{n+1} : \langle \mathbf{u}_i, \mathbf{z} \rangle \leq {\rm val}(c_{i,\mathbf{a}}) + \langle \mathbf{a}, \mathbf{z} \rangle, \text{ for } 1 \leq i \leq s, \mathbf{a} \in \mathbb{N}^{n+1} \, \},$$

then $\overline{C_I[\mathbf{w}]}$ is a $\Gamma_{\rm val}$-rational polyhedron.

A polyhedron $P = \{ \, \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b} \, \}$ is Γ-rational if $A \in \mathbb{Q}^{d \times n}$ and $\mathbf{b} \in \Gamma^d$.

proof of the inequality description

Proposition (Proposition 2.5.2)

The set $\overline{C_I[\mathbf{w}]}$ is a Γ-rational polyhedron which contains the line $\mathbb{R}(1, 1, \ldots, 1)$ as its largest affine subspace.
If $\mathrm{in}_{\mathbf{w}}(I)$ is not a monomial ideal, then there exists $\mathbf{w}' \in \Gamma_{\mathrm{val}}^{n+1}$ such that $\mathrm{in}_{\mathbf{w}'}(I)$ is a monomial ideal and $\overline{C_I[\mathbf{w}]}$ is a proper face of $\overline{C_I[\mathbf{w}']}$.

Proof:

- By Lemma 2.4.6, $\exists \mathbf{v} \in \mathbb{Q}^{n+1}$, $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is a monomial ideal.
- By Corollary 2.4.9: $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ for $\epsilon > 0$.
  Fix such ϵ, let $\mathbf{w}' = \mathbf{w} + \epsilon\mathbf{v}$ and $\mathrm{in}_{\mathbf{w}'}(I) = \langle \mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2}, \ldots, \mathbf{x}^{\mathbf{u}_s} \rangle$.
- By Lemma 2.4.7, the monomials not in $\mathrm{in}_{\mathbf{w}'}(I)$ of degree $d = \deg(\mathbf{x}^{\mathbf{u}_i})$ form a basis for $(S/I)_d$.

proof continued

$\exists \mathbf{v} \in \mathbb{Q}^{n+1}$, $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is a monomial ideal, $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ for $\epsilon > 0$, $\mathbf{w}' = \mathbf{w} + \epsilon\mathbf{v}$, $\mathrm{in}_{\mathbf{w}'}(I) = \langle \mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2}, \ldots, \mathbf{x}^{\mathbf{u}_s} \rangle$. The monomials not in $\mathrm{in}_{\mathbf{w}'}(I)$ of degree $d = \deg(\mathbf{x}^{\mathbf{u}_i})$ form a basis for $(S/I)_d$.

- Let g_i' be the result of writing $\mathbf{x}^{\mathbf{u}_i}$ in this basis, so no monomial occurring in g_i' lies in $\mathrm{in}_{\mathbf{w}'}(I)$.
- We write $c_{i\mathbf{v}}$ for the coefficient of $\mathbf{x}^{\mathbf{v}}$ in g_i'.
- The polynomial $g_i = \mathbf{x}^{\mathbf{u}_i} - g_i'$ is in I.
- Since $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(g_i))$ must lie in $\mathrm{in}_{\mathbf{w}'}(I)$, we have $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(g_i)) = \mathbf{x}^{\mathbf{u}_i}$, and thus $\mathrm{in}_{\mathbf{w}'}(g_i) = \mathbf{x}^{\mathbf{u}_i}$.
- The polynomials $\{g_1, g_2, \ldots, g_s\}$ form a Gröbner basis for I with respect to $\mathbf{w}'$.

the inequality description

For $C_I[\mathbf{w}'] = \{ \mathbf{w}'' \in \Gamma_{\rm val}^{n+1} : {\rm in}_{\mathbf{w}''}(I) = {\rm in}_{\mathbf{w}'}(I) \}$, we prove that $\overline{C_I[\mathbf{w}']}$ is

$$P = \{ \mathbf{z} \in \mathbb{R}^{n+1} : \langle \mathbf{u}_i, \mathbf{z} \rangle \leq {\rm val}(c_{i,\mathbf{a}}) + \langle \mathbf{a}, \mathbf{z} \rangle, \text{ for } 1 \leq i \leq s, \mathbf{a} \in \mathbb{N}^{n+1} \}.$$

Suppose $\widetilde{\mathbf{w}} \in C_I[\mathbf{w}']$,

- but one of the inequalities $\langle \mathbf{u}_i, \mathbf{z} \rangle \leq {\rm val}(c_{i,\mathbf{a}}) + \langle \mathbf{a}, \mathbf{z} \rangle$ is violated.
- For that index i, we have ${\rm in}_{\widetilde{\mathbf{w}}}(g_i) \neq \mathbf{x}^{\mathbf{u}_i}$.
- Since ${\rm in}_{\mathbf{w}'}(I) = {\rm in}_{\widetilde{\mathbf{w}}}(I)$ is a monomial ideal,
  every term of ${\rm in}_{\widetilde{\mathbf{w}}}(g_i)$ lies in ${\rm in}_{\widetilde{\mathbf{w}}}(I)$,

which contradicts the construction of g_i. Thus $\overline{C_I[\mathbf{w}']} \subseteq P$.

the reverse inclusion

For $C_I[\mathbf{w}'] = \{ \mathbf{w}'' \in \Gamma_{\mathrm{val}}^{n+1} : \mathrm{in}_{\mathbf{w}''}(I) = \mathrm{in}_{\mathbf{w}'}(I) \}$, to show that $\overline{C_I[\mathbf{w}']}$ contains

$$P = \{ \mathbf{z} \in \mathbb{R}^{n+1} : \langle \mathbf{u}_i, \mathbf{z} \rangle \leq \mathrm{val}(c_{i,\mathbf{a}}) + \langle \mathbf{a}, \mathbf{z} \rangle, \text{ for } 1 \leq i \leq s, \mathbf{a} \in \mathbb{N}^{n+1} \},$$

assume $\langle \mathbf{u}_i, \widetilde{\mathbf{w}} \rangle < \mathrm{val}(c_{i,\mathbf{a}}) + \langle \mathbf{a}, \widetilde{\mathbf{w}} \rangle$, for all i.

- Then $\mathrm{in}_{\widetilde{\mathbf{w}}}(g_i) = \mathbf{x}^{\mathbf{u}_i}$ for all i,
- and hence: $\mathrm{in}_{\widetilde{\mathbf{w}}}(I) \subseteq \mathrm{in}_{\mathbf{w}'}(I)$.
- The two ideals have the same Hilbert function, so they are equal.

We conclude $\widetilde{\mathbf{w}} \in C_I[\mathbf{w}']$.

$\overline{C_I[\mathbf{w}]}$ is a proper face of $\overline{C_I[\mathbf{w}']}$

Recall $\mathbf{w}' = \mathbf{w} + \epsilon\mathbf{v}$, $\mathrm{in}_{\mathbf{w}'}(I) = \langle \mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2}, \ldots, \mathbf{x}^{\mathbf{u}_s} \rangle$, and $\{g_1, g_2, \ldots, g_s\}$ forms a Gröbner basis for I with respect to $\mathbf{w}'$, so $\mathrm{in}_{\mathbf{w}'}(g_i) = \mathbf{x}^{\mathbf{u}_i}$.
This shows $C_I[\mathbf{w}] \subset \overline{C_I[\mathbf{w}']}$.

$\overline{C_I[\mathbf{w}]}$ being a Γ_{val}-polyhedron is implied by being a face of $\overline{C_I[\mathbf{w}']}$.

Note that $\{\ \mathrm{in}_{\mathbf{w}}(g_1), \mathrm{in}_{\mathbf{w}}(g_2), \ldots, \mathrm{in}_{\mathbf{w}}(g_s)\ \}$ is a Gröbner basis for $\mathrm{in}_{\mathbf{w}}(I)$ with respect to $\mathbf{v}$. If $\widetilde{\mathbf{w}} \in \Gamma_{\mathrm{val}}^{n+1}$ satisfies $\mathrm{in}_{\widetilde{\mathbf{w}}}(I) = \mathrm{in}_{\mathbf{w}}(I)$, then $\mathrm{in}_{\widetilde{\mathbf{w}}}(g_i) = \mathrm{in}_{\mathbf{w}}(g_i)$, for all i. Otherwise, $\mathrm{in}_{\widetilde{\mathbf{w}}}(g_i)$ would still have $\mathbf{x}^{\mathbf{u}_i}$ in its support or $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\widetilde{\mathbf{w}}}(I))$ would not be equal to the monomial ideal $\mathrm{in}_{\mathbf{w}'}(I)$.

But then $\mathrm{in}_{\widetilde{\mathbf{w}}}(g_i) - \mathrm{in}_{\mathbf{w}}(g_i) \in \mathrm{in}_{\mathbf{w}}(I)$, and this polynomial does not contain any monomials from $\mathrm{in}_{\mathbf{w}'}(I)$, contradicting $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}'}(I)$.

We conclude that $\overline{C_I[\mathbf{w}]}$ is the set of points $\mathbf{z}$ in the cone $\overline{C_I[\mathbf{w}']}$ that satisfy $\langle \mathbf{u}_i, \mathbf{z} \rangle = \mathrm{val}(c_{i,\mathbf{a}}) + \langle \mathbf{a}, \mathbf{z} \rangle$ whenever $\mathbf{x}^{\mathbf{a}}$ appears in $\mathrm{in}_{\mathbf{w}}(g_i)$.
So $\overline{C_I[\mathbf{w}]}$ is a face of $\overline{C_I[\mathbf{w}']}$.

the lineality space $\mathbb{R}\mathbf{1} = \mathbb{R}(1, 1, \ldots, 1)$

Finally, for any homogeneous polynomial $f \in K[x_0, x_1, \ldots, x_n]$ we have $\mathrm{in}_{\mathbf{w}}(f) = \mathrm{in}_{\mathbf{w}+\lambda\mathbf{1}}(f)$ for all $\lambda \in \Gamma_{\mathrm{val}}$.

Since all initial ideals of I are generated by homogeneous polynomials, by Lemma 2.4.2, this implies $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{w}+\lambda\mathbf{1}}(I)$ for all $\lambda \in \Gamma_{\mathrm{val}}$.

Therefore, $\overline{C_I[\mathbf{w}]} = \overline{C_I[\mathbf{w}]} + \mathbb{R}\mathbf{1}$.

The lineality space of the polyhedron $\overline{C_I[\mathbf{w}]}$ contains the line $\mathbb{R}\mathbf{1}$.