Algebraic Varieties and Polyhedral Geometry

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Graduate Computational Algebraic Geometry Seminar

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Introduction to Tropical Geometry

Introduction to Tropical Geometry is the title of a forthcoming book of Diane Maclagan and Bernd Sturmfels.

The web page
`http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.html`
offers the pdf file of a book, dated 28 February 2014.

Today we look at some building blocks ...

This seminar is based on sections 1.4, 2.2, and 2.4.

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Zalessky's conjecture

- $S = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ Laurent polynomial ring
- $g = (g_{i,j}) \in \mathrm{GL}(n, \mathbb{Z})$ invertible integer matrix defines the action

$$g : S \rightarrow S : x_i \mapsto \prod_{j=1}^{n} x_j^{g_{i,j}}$$

- I is a proper ideal in S
- the stabilizer group of I is

$$\mathrm{Stab}(I) = \{ \, g \in \mathrm{GL}(n, \mathbb{Z}) : gI = I \, \}$$

Theorem (theorem 1 of Bergman 1971)

$\mathrm{Stab}(I)$ *has a subgroup of finite index,*
which stabilizes a nontrivial sublattice of $\mathbb{Z}^n$.

from the paper of George M. Bergman

Theorem (theorem 1 of Bergman 1971)

Let I be a nontrivial ideal in $K[\mathbf{x}^{\pm 1}]$, and $H \subseteq \mathrm{GL}(n, \mathbb{Z})$ the stabilizer subgroup of I. Then H has a subgroup H_0 of finite index, which stabilizes a nontrivial subgroup of $\mathbb{Z}^n$ (equivalently, which can be put into block-triangular form

$$\left(\begin{array}{c|c} \star & \star \\ \hline 0 & \star \end{array} \right)$$

in $\mathrm{GL}(n, \mathbb{Z})$).

Bergman's conceptual proof for $K = \mathbb{C}$

Consider $V \subseteq (\mathbb{C} \setminus \{0\})^n$ defined by some nontrivial ideal.

1. Look at limiting values of ratios $\log|x_1| : \log|x_2| : \cdots : \log|x_n|$ as $\mathbf{x} \in V$ becomes large.
   Identify this set of ratios with the $(n-1)$-sphere S^{n-1}.
2. The limiting ratios of logarithms lies in a finite union of proper great subspheres on S^{n-1}, having rational defining parameters.
3. Assuming this, note:
   - the intersection of two such finite unions of subspheres will again be one;
   - the family of all finite unions of great subspheres has a descending chain condition.

   There exists a unique finite union U of subspheres minimal for the property of containing all "logarithmic limit-points at infinity" of V.
   If V has positive dimension, U must be nonempty.

the proof continued

4. The space of our n-tuples of logarithms $\mathbb{R}^n$ arises as the dual of $\mathbb{Z}^n$, that is: $\mathrm{Hom}_{\mathrm{groups}}(\mathbb{Z}^n, \mathbb{R})$.
   Thus we get a natural action of $\mathrm{GL}(n, \mathbb{Z})$ on $\mathbb{R}^n$, and so on S^{n-1}.
5. Clearly U will be invariant under the induced action of the stabilizer subgroup, H, of I.
   By duality, we obtain from the great subspheres of U a family Q of nontrivial subgroups of $\mathbb{Z}^n$, also invariant under H.

Q.E.D.

The claim that logarithmic points at infinity of V lie in a finite union of proper great subspheres of S^{n-1}, consider the support A of any nonzero $f \in I$. At $\mathbf{z} \in V$: $f(\mathbf{z}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{z}^{\mathbf{a}} = 0$.

At each point of V, at least two terms of the sum (the largest ones) must be of the same order of magnitude.
Each $\log |\mathbf{z}|$ lies in one of the finite family of “planks” in $\mathbb{R}^n$.

a lemma

Denote the standard unit vectors by $\mathbf{e}_1$, $\mathbf{e}_2$, $\ldots$

Lemma (Lemma 2.2.9)

1. *Given any* $\mathbf{v} \in \mathbb{Z}^n$ *with* $\gcd(|v_1|, |v_2|, \ldots, |v_n|) = 1$*.*
   There is a matrix $U \in \mathrm{GL}(n, \mathbb{Z})$*:* $U\mathbf{v} = \mathbf{e}_1$*.*
2. *Let* L *be a rank* k *subgroup of* $\mathbb{Z}^n$ *with* $\mathbb{Z}^n/L$ *torsion-free.*
   There is a matrix $U \in \mathrm{GL}(n, \mathbb{Z})$ *with* UL *equal to the subgroup generated by* $\mathbf{e}_1$*,* $\mathbf{e}_2$*,* $\ldots$*,* $\mathbf{e}_k$*.*

To prove the first statement:

$$
\begin{array}{rcl}
1 & = & \gcd(v_1, v_2) \\
  & = & a v_1 + b v_2
\end{array}
\qquad
\left[\begin{array}{cc} a & b \\ -v_2 & v_1 \end{array} \right]
\left[\begin{array}{c} v_1 \\ v_2 \end{array} \right]
=
\left[\begin{array}{c} 1 \\ 0 \end{array} \right].
$$

Apply $n - 1$ times repeatedly for a vector of length n.

torsion-free

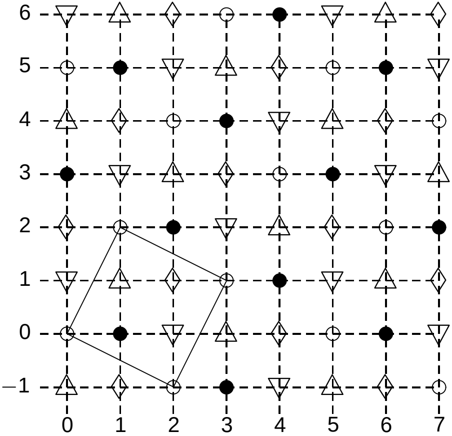
$\mathbb{Z}$-module: like a vector space we have scalar multiplication, but $\mathbb{Z}$ is a ring, not a field.

A group G is torsion-free if

$$\forall g \in G \setminus \{0\} \text{ and } \forall n \in \mathbb{Z} \setminus \{0\} : ng \neq 0.$$

For $n \in \mathbb{Z} \setminus \{0\}$: $\mathbb{Z}/n\mathbb{Z}$ is not torsion-free.

an example of a lattice



$$L = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad G = \mathbb{Z}^2 / L = \langle g_1, g_2 \rangle \Big/ \begin{pmatrix} 2g_1 - g_2 = 0 \\ g_1 + 2g_2 = 0 \end{pmatrix} \simeq \mathbb{Z}/5\mathbb{Z}$$

proof of Lemma 2.2.9

Let L be a rank k subgroup of $\mathbb{Z}^n$ with $\mathbb{Z}^n/L$ torsion-free. There is a matrix $U \in \mathrm{GL}(n,\mathbb{Z})$ with UL equal to the subgroup generated by $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_k$.

Let $A \in \mathbb{Z}^{k \times n}$ contains in its rows a basis for L.

$\mathbb{Z}^n/L$ is torsion-free $\Rightarrow$ Smith Normal Form (SNF) of A is $A' = [I \;\; 0]$, where I is the identity matrix.

By SNF: $A' = VAU'$, for $V \in \mathrm{GL}(k,\mathbb{Z})$ and $U' \in \mathrm{GL}(n,\mathbb{Z})$.

Because multiplication by invertible matrix does not change row span, the row span of VA is the same as the row span of L.

$$A' = [I \;\; 0] = [\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \cdots \;\; \mathbf{e}_k]^T = (VA)U'$$

As $A'^T = U'^T(VA)^T$, take $U = U'^T$.

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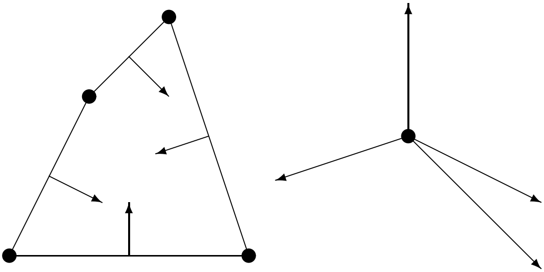
inner normal fans

Consider a Newton polygon with inner normals to its edges:



The inner normal fan is shown at the left:

- the rays are normal to the edges of the polygon;
- normals to the vertices of the polygon are contained in the strict interior of cones spanned by the rays.

polyhedral fans

Let P be an n-dimensional polytope.
Denote the inner product by $\langle \cdot, \cdot \rangle$.

For $\mathbf{v} \neq 0$, the *face of P defined by $\mathbf{v}$* is

$$\mathrm{in}_{\mathbf{v}}(P) = \{ \mathbf{a} \in P \mid \langle \mathbf{a}, \mathbf{v} \rangle = \min_{\mathbf{b} \in P} \langle \mathbf{b}, \mathbf{v} \rangle \}.$$

The $\mathrm{in}_{\mathbf{v}}(\cdot)$ notation refers to inner forms of polynomials that are supported on faces of the Newton polytopes.

If we have a face F of P, then its *inner normal cone* is

$$\mathrm{cone}(F) = \{ \mathbf{v} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{v}}(P) = F \}.$$

Passing from a face to its normal cone is like passing to the dual.
Taking the dual of the dual brings us back to the original.

Minkowski sum and common refinement

The *Minkowski sum* of two sets $A, B \subset \mathbb{R}^n$:

$$A + B = \{ \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B \}.$$

The Newton polytope of the product of two polynomials is the Minkowski sum of their Newton polytopes.

The *common refinement* of two polyhedral fans $\mathcal{F}$ and $\mathcal{G}$ is

$$\mathcal{F} \wedge \mathcal{G} = \{ P \cap Q \mid P \in \mathcal{F}, Q \in \mathcal{Q} \}.$$

The normal fan of the Minkowski sum of two polytopes is the common refinement of their normal fans.

regular subdivisions

Let $P = \mathrm{conv}(\mathbf{a}_i, i = 1, 2, \ldots, r) \subset \mathbb{R}^n$.

A *regular subdivision* of P is induced by $\mathbf{w} = (w_1, w_2, \ldots, w_r)$:

1. $\widehat{P} = \mathrm{conv}((\mathbf{a}_i, w_i) \mid i = 1, 2, \ldots, r)$.
2. Projecting the facets on the lower hull of $\widehat{P}$ onto $\mathbb{R}^n$ — dropping the last coordinate — gives the cells in the regular subdivision induced by $\mathbf{w}$.

If all cells are simplices (spanned by exactly $n + 1$ points), then the regular subdivision is a regular triangulation.

A *polyhedral complex* $\mathcal{C}$ is a collection of polyhedra:

1. If a polyhedron $P \in \mathcal{C}$, then for all $\mathbf{v}$: $\mathrm{in}_{\mathbf{v}}(P) \in \mathcal{C}$.
2. If $P, Q \in \mathcal{C}$, then either $P \cap Q = \emptyset$ or $P \cap Q$ is a face of both.

Polytopes, fans, and subdivisions are polyhedral complexes.

algorithms and software

The computation of the convex hull is a major problem solved by computational geometry. Problem specification:

- a collection of points in the plane or in space;
- a description of all faces of the convex hull.

Solution: apply the beneath-beyond or the giftwrapping method.
Software: Qhull.

In optimization, the linear programming method solves

$$\begin{array}{l} \min \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to } A\mathbf{x} \geq b \end{array}$$

Inner normals to facets are subject to a system of linear inequalities.
Software: cddlib, lrs.

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the setup

- K : coefficient field, not required to be algebraically closed
- S : the polynomial ring $S = K[x_0, x_1, \ldots, x_n]$
- I : a homogeneous ideal in S
- val : a nontrivial valuation, $\mathrm{val} : K \rightarrow \mathbb{R} \cup \{\infty\}$
- R : the valuation ring of K, $R = \mathrm{val}(K^*), K^* = K \setminus \{0\}$
- Γ_{val} : the value group is dense in $\mathbb{R}$, $\Gamma_{\mathrm{val}} = \{ x \in K : \mathrm{val}(x) \geq 0 \}$
  $\Gamma_{\mathrm{val}} = \mathbb{Q}$ for Puiseux series $\mathbb{C}\{\{t\}\}[\mathbf{x}^{\pm \mathbf{1}}]$
- $\mathbb{K}$: the residue field, $\mathbb{K} = R/\mathbf{m}$, $\mathbf{m} = \{ x \in K : \mathrm{val}(x) > 0 \}$
  If $c \in K$, then denote $\overline{c} \in \mathbb{K}$.
  For polynomials $f \in S$:

$$f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, c_{\mathbf{a}} \in K^* \qquad \overline{f} = \sum_{\mathbf{a} \in A} \overline{c_{\mathbf{a}}} \mathbf{x}^{\mathbf{a}}.$$

initial forms

The *tropicalization* of $f = \sum_{\mathbf{a} \in A} c_\mathbf{a} \mathbf{x}^\mathbf{a}$ is a piecewise linear function

$$\mathrm{trop}(f) : \mathbb{R}^{n+1} \rightarrow \mathbb{R} : \mathbf{w} \mapsto \mathrm{trop}(f)(\mathbf{w}) = \min(\mathrm{val}(c_\mathbf{a}) + \langle \mathbf{a}, \mathbf{w} \rangle, \mathbf{a} \in A).$$

The *initial form* of f with respect to $\mathbf{w}$ is

$$\begin{aligned}
\mathrm{in}_\mathbf{w}(f) &= \overline{t^{-\mathrm{trop}(f)(\mathbf{w})} f(t^{w_0} x_0, t^{w_1} x_1, \ldots, t^{w_n} x_n)} \\
&= \overline{t^{-W} \sum_{\mathbf{a} \in A} c_\mathbf{a} t^{\langle \mathbf{a}, \mathbf{w} \rangle} \mathbf{x}^\mathbf{a}}, \quad W = \mathrm{trop}(f)(\mathbf{w}) \\
&= \sum_{\substack{\mathbf{a} \in A \\ \mathrm{val}(c_\mathbf{a}) + \langle \mathbf{a}, \mathbf{w} \rangle = W}} \overline{c_\mathbf{a} t^{-\mathrm{val}(c_\mathbf{a})}} \mathbf{x}^\mathbf{a} \\
&\in \mathbb{K}[x_0, x_1, \ldots, x_n].
\end{aligned}$$

an example

$$f = (t + t^2)x_0 + 2t^2 x_1 + 3t^4 x_2 \in \mathbb{C}\{\{t\}\}[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]$$

$c(t) \in \mathbb{C}\{\{t\}\}$, $c(t) = t^{b_1}(1 + O(t))$: $\mathrm{val}(c(t)) = b_1$

$W = \min(\mathrm{val}(c_{\mathbf{a}}) + \langle \mathbf{a}, \mathbf{w} \rangle, \mathbf{a} \in A)$

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\substack{\mathbf{a} \in A \\ \mathrm{val}(c_{\mathbf{a}}) + \langle \mathbf{a}, \mathbf{w} \rangle = W}} \overline{c_{\mathbf{a}} t^{-\mathrm{val}(c_{\mathbf{a}})}} \mathbf{x}^{\mathbf{a}}$$

- If $\mathbf{w} = (0,0,0)$, then $W = 1$ and $\mathrm{in}_{\mathbf{w}}(f) = \overline{(1+t)x_0} = x_0$.
- If $\mathbf{w} = (4,2,0)$, then $W = 4$ and $\mathrm{in}_{\mathbf{w}}(f) = 2x_1 + 3x_2$.

  Note: $\mathrm{in}_{(2,1,0)}(f) = x_0 + 2x_1$.

initial ideals and Gröbner bases

The *initial ideal* of a homogeneous ideal I in S is

$$\mathrm{in}_{\mathbf{w}}(I) = \langle\ \mathrm{in}_{\mathbf{w}}(f)\ :\ f \in I\ \rangle \subset \mathbb{K}[x_0, x_1, \ldots, x_n].$$

A *Gröbner basis* for I with respect to **w** is

- a finite set $\mathcal{G} = \{\ g_1, g_2, \ldots, g_s\ \} \subset I$,
- with $\langle\ \mathrm{in}_{\mathbf{w}}(g_1), \mathrm{in}_{\mathbf{w}}(g_2), \ldots, \mathrm{in}_{\mathbf{w}}(g_s)\ \rangle = \mathrm{in}_{\mathbf{w}}(I)$.

Lemma (Lemma 2.4.2)

Let $I \subset K[x_0, x_1, \ldots, x_n]$ be a homogeneous ideal and fix $\mathbf{w} \in (\Gamma_{\mathrm{val}})^{n+1}$.
Then $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous and we may choose a homogeneous Gröbner basis for I.
Furthermore, if $g \in \mathrm{in}_{\mathbf{w}}(I)$, then $g = \mathrm{in}_{\mathbf{w}}(f)$ for some $f \in I$.

proof of the lemma

To see $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous, consider $f = \sum_{i \geq 0} f_i \in S$,
where $\deg(f_i) = i$ and f_i is homogeneous.

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\substack{i \geq 0 \\ \mathrm{trop}(f)(\mathbf{w}) = \mathrm{trop}(f_i)(\mathbf{w})}} \mathrm{in}_{\mathbf{w}}(f_i)$$

Since each homogeneous component of f_i lives in I,
$\mathrm{in}_{\mathbf{w}}(I)$ is generated by elements $\mathrm{in}_{\mathbf{w}}(f)$ with f homogeneous.

The initial form of a homogeneous polynomial is homogeneous,
so this means that $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous.

As S is Noetherian, $\mathrm{in}_{\mathbf{w}}(I)$ is generated by a finite number of these $\mathrm{in}_{\mathbf{w}}(f)$, so the corresponding f form a Gröbner basis for I.

proof of the last claim in the lemma

Furthermore, if $g \in \mathrm{in}_{\mathbf{w}}(I)$, then $g = \mathrm{in}_{\mathbf{w}}(f)$ for some $f \in I$.

$$g = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} x^{\mathbf{a}} \mathrm{in}_{\mathbf{w}}(f_{\mathbf{a}}) \in \mathrm{in}_{\mathbf{w}}(I), \quad \text{with } f_{\mathbf{a}} \in I, \text{ for all } \mathbf{a}$$

Then $g = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathrm{in}_{\mathbf{w}}(\mathbf{x}^{\mathbf{a}} f_{\mathbf{a}})$.

- For each $c_{\mathbf{a}}$, choose a lift $r_{\mathbf{a}} \in R$ with $\mathrm{val}(r_{\mathbf{a}}) = 0$ and $\overline{r_{\mathbf{a}}} = c_{\mathbf{a}}$.
- Let $W_{\mathbf{a}} = \mathrm{trop}(f_{\mathbf{a}})(\mathbf{w}) + \langle \mathbf{w}, \mathbf{a} \rangle$.
- Let $f = \sum_{\mathbf{a} \in A} r_{\mathbf{a}} t^{-W_{\mathbf{a}}} \mathbf{x}^{\mathbf{a}} f_{\mathbf{a}}$.

Then, by construction, $\mathrm{trop}(f)(\mathbf{w}) = 0$ and $\mathrm{in}_{\mathbf{w}}(f) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \mathrm{in}_{\mathbf{w}}(f) = g$.