Gröbner Complexes and Tropical Bases

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Graduate Computational Algebraic Geometry Seminar

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Introduction to Tropical Geometry

Introduction to Tropical Geometry is the title of a forthcoming book of Diane Maclagan and Bernd Sturmfels.

The web page
`http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.html`
offers the pdf file of a book, dated 31 March 2014.

Today we look at some building blocks ...

This seminar is based on sections 2.4, 2.5, and 2.6.

Gröbner Complexes and Tropical Bases

a Gröbner basis that does not generate the ideal

The *initial ideal* of a homogeneous ideal I in $K[x_0, x_1, \ldots, x_n]$ is $\mathrm{in}_{\mathbf{w}}(I) = \langle\ \mathrm{in}_{\mathbf{w}}(f)\ :\ f \in I\ \rangle \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$, $\mathbb{K}$ is the residue field.
A *Gröbner basis* for I with respect to $\mathbf{w}$ is

- a finite set $\mathcal{G} = \{\ g_1, g_2, \ldots, g_s\ \} \subset I$,
- with $\langle\ \mathrm{in}_{\mathbf{w}}(g_1), \mathrm{in}_{\mathbf{w}}(g_2), \ldots, \mathrm{in}_{\mathbf{w}}(g_s)\ \rangle = \mathrm{in}_{\mathbf{w}}(I)$.

Consider a ***nonhomogeneous*** ideal $I = \langle x \rangle \subset K[x]$ and $w = 1$.
Then $\mathcal{G} = \{x - x^2\}$ is a Gröbner basis for I,
as $\mathrm{in}_{(1)}(\mathcal{G}) = \{x\}$ and $\mathrm{in}_{(1)}(I) = \langle x \rangle$.

However, $\mathcal{G}$ does not generate I because $x \not\in \langle x - x^2 \rangle$.
There is no polynomial $f \in K[x]$: $x = (x - x^2)f$.

$$
\begin{aligned}
x = (x - x^2)f \quad & \Leftrightarrow \quad 1 = (1 - x)f \\
& \Leftrightarrow \quad 1 \cdot x^0 + 0 \cdot x^1 = f \cdot x^0 - f \cdot x^1 \\
& \Leftrightarrow \quad 1 = f \text{ and } 0 = f.
\end{aligned}
$$

local orders and homogeneous ideals

The point is that $\mathrm{in}_{\mathbf{w}}$ is a local, not a global monomial order.

Consider

$$(1-x)x = x - x^2 \quad \Leftrightarrow \quad x = \frac{x-x^2}{1-x} = (x-x^2)\sum_{i=0}^{\infty} x^i.$$

Instead of working with power series, a *weak* normal form is computed by Mora's normal form algorithm, which leads to standard basis.

For homogeneous ideals, Mora's normal form algorithm becomes equal to Buchberger's algorithm to compute a Gröbner basis.

Working with homogeneous ideals I over fields with valuations, the definition $\langle\, \mathrm{in}_{\mathbf{w}}(g_1), \mathrm{in}_{\mathbf{w}}(g_2), \ldots, \mathrm{in}_{\mathbf{w}}(g_s) \,\rangle = \mathrm{in}_{\mathbf{w}}(I)$ for a finite set $\mathcal{G} = \{\, g_1, g_2, \ldots, g_s \,\} \subset I$ gives a basis for I.

initial forms of initial forms of polynomials

The initial form of an initial form is an initial form.

Lemma (Lemma 2.4.5)

Fix $f \in K[x_0, x_1, \ldots, x_n]$, $\mathbf{w} \in \Gamma_{\rm val}^{n+1}$, *and* $\mathbf{v} \in \mathbb{Q}^{n+1}$.
There exists an $\epsilon > 0$ *such that for all* $\delta \in \Gamma_{\rm val}$ *with* $0 < \delta < \epsilon$, *we have*

$$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f)) = \mathrm{in}_{\mathbf{w}+\delta\mathbf{v}}(f).$$

Proof. Let $f = \displaystyle\sum_{\mathbf{u} \in \mathbb{N}^{n+1}} c_{\mathbf{u}} x^{\mathbf{u}}$. $W = \min(\mathrm{val}(c_{\mathbf{u}}) + \langle \mathbf{u}, \mathbf{w} \rangle, c_{\mathbf{u}} \neq 0)$. Then:

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\mathbf{u} \in \mathbb{N}^{n+1}} \overline{c_{\mathbf{u}} t^{\langle \mathbf{w}, \mathbf{u} \rangle - W}} \mathbf{x}^{\mathbf{u}}.$$

Let $W' = \min(\langle \mathbf{v}, \mathbf{u} \rangle : \mathrm{val}(c_{\mathbf{u}}) + \langle \mathbf{u}, \mathbf{w} \rangle = W)$. Then:

$$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f)) = \sum_{\langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{W}'} \overline{c_{\mathbf{u}} t^{\langle \mathbf{w}, \mathbf{u} \rangle - W}} \mathbf{x}^{\mathbf{u}}.$$

proof continued

Recall the notations:

$$\begin{aligned} W &= \min(\mathrm{val}(c_{\mathbf{u}}) + \langle \mathbf{u}, \mathbf{w} \rangle, c_{\mathbf{u}} \neq 0) = \mathrm{trop}(f)(\mathbf{w}), \\ W' &= \min(\langle \mathbf{v}, \mathbf{u} \rangle : \mathrm{val}(c_{\mathbf{u}}) + \langle \mathbf{u}, \mathbf{w} \rangle = W). \end{aligned}$$

For sufficiently small $\epsilon > 0$, we have:

$$\mathrm{trop}(f)(\mathbf{w} + \epsilon \mathbf{v}) = \min(\mathrm{val}(c_{\mathbf{u}}) + \langle \mathbf{w}, \mathbf{u} \rangle + \epsilon \langle \mathbf{v}, \mathbf{u} \rangle) = W + \epsilon W'$$

and

$$\begin{aligned} & \{ \mathbf{u} : \mathrm{val}(c_{\mathbf{u}}) + \langle \mathbf{w} + \delta \mathbf{v}, \mathbf{u} \rangle = W + \epsilon W' \} \\ = & \{ \mathbf{u} : \mathrm{val}(c_{\mathbf{u}}) + \langle \mathbf{w}, \mathbf{u} \rangle = W, \langle \mathbf{v}, \mathbf{u} \rangle = W' \}. \end{aligned}$$

This implies $\mathrm{in}_{\mathbf{w}+\delta\mathbf{v}}(f) = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f))$ for $\delta \in \Gamma_{\mathrm{val}}$ with $0 < \delta < \epsilon$. QED

initial forms of initial forms of ideals

Lemma (Lemma 2.4.6)

Let I be a homogeneous ideal in $K[x_0, x_1, \ldots, x_n]$ and fix $\mathbf{w} \in \Gamma_{\rm val}$.
There exists a $\mathbf{v} \in \mathbb{Q}^{n+1}$ and $\epsilon > 0$ such that

1. $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ *and* $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$ *are monomial ideals; and*
2. $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$.

Corollary (Corollary 2.4.9)

Let I be a homogeneous ideal in $K[x_0, x_1, \ldots, x_n]$.
For any $\mathbf{w} \in \Gamma_{\rm val}^{n+1}$ and $\mathbf{v} \in \mathbb{Q}^{n+1}$ there exists $\epsilon > 0$ such that

$$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}+\delta\mathbf{v}}(I) \quad \textit{for all } 0 < \delta < \epsilon \textit{ with } \delta\mathbf{v} \in \Gamma_{\rm val}^{n+1}.$$

Gröbner Complexes and Tropical Bases

motivation

For homogeneous ideals in $K[x_0, x_1, \ldots, x_n]$ over a field with a valuation we can define Gröbner bases.

There is no natural intrinsic notion for Gröbner bases for ideals in the Laurent polynomial ring $K[\mathbf{x}^{\pm \mathbf{1}}] = K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

The tropical basis is the natural analogue to the notion of a *universal* Gröbner basis.

The goal of section 2.5 is to construct a polyhedral complex from a given homogeneous ideal $I \subset K[x_0, x_1, \ldots, x_n]$.

a universal Gröbner basis

Consider a homogeneous ideal $I \subset K[x_0, x_1, \ldots, x_n]$.
A universal Gröbner basis for I

- is a finite subset $\mathcal{U}$ of I, such that:
- for all $\mathbf{w} \in \Gamma_{\mathrm{val}}^{n+1}$, $\mathrm{in}_{\mathbf{w}}(\mathcal{U}) = \{ \mathrm{in}_{\mathbf{w}}(f) : f \in \mathcal{U} \}$ generates $\mathrm{in}_{\mathbf{w}}(I)$ in $\mathbb{K}[x_0, x_1, \ldots, x_n]$.

Example:

$$\mathcal{U} = \left\{ \begin{array}{c} x_1(x_2 + x_3 - x_1), x_2(x_1 + x_3 - x_2), x_3(x_1 + x_2 - x_3), \\ x_1x_2(x_1 - x_2), x_1x_3(x_1 - x_3), x_2x_3(x_2 - x_3) \end{array} \right\}.$$

is a universal Gröbner basis for

$$I = \langle \mathcal{U} \rangle = \langle x_1 - x_2, x_3 \rangle \cap \langle x_1 - x_3, x_2 \rangle \cap \langle x_2 - x_3, x_1 \rangle.$$

However, I contains $x_1x_2x_3$, a unit in $K[\mathbf{x}^{\pm 1}]$, so $\mathcal{U}$ is not tropical basis.

defining polyhedra

For a homogeneous ideal $I \subset K[x_0, x_1, \ldots, x_n]$ and for $\mathbf{w} \in \Gamma_{\rm val}^{n+1}$ we set

$$C_I[\mathbf{w}] = \{ \, \mathbf{v} \in \Gamma_{\rm val}^{n+1} : {\rm in}_{\mathbf{v}}(I) = {\rm in}_{\mathbf{w}}(I) \, \}.$$

Let $\overline{C_I[\mathbf{w}]}$ be the closure of $C_I[\mathbf{w}]$ in $\mathbb{R}^{n+1}$ in the Euclidean topology.

Consider a Gröbner basis $\{g_1, g_2, \ldots, g_s\}$ of I with respect to $\mathbf{w}$, and let ${\rm in}_{\mathbf{w}}(g_i) = \mathbf{x}^{\mathbf{u}_i}$, for $g_i = \displaystyle\sum_{\mathbf{v} \in \mathbb{N}^{n+1}} c_{i,\mathbf{v}} \mathbf{x}^{\mathbf{v}}$.

If $\overline{C_I[\mathbf{w}]}$ has the inequality description

$$\{ \, \mathbf{z} \in \mathbb{R}^{n+1} : \langle \mathbf{u}_i, \mathbf{z} \rangle \leq {\rm val}(c_{i,\mathbf{v}}) + \langle \mathbf{v}, \mathbf{z} \rangle, \text{ for } 1 \leq i \leq s, \mathbf{v} \in \mathbb{N}^{n+1} \, \},$$

then $\overline{C_I[\mathbf{w}]}$ is a $\Gamma_{\rm val}$-rational polyhedron.

A polyhedron $P = \{ \, \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b} \, \}$ is Γ-rational if $A \in \mathbb{Q}^{d \times n}$ and $\mathbf{b} \in \Gamma^d$.

the inequality description

The inequality description of $\overline{C_I[\mathbf{w}]}$

$$\{\ \mathbf{z} \in \mathbb{R}^{n+1} : \langle \mathbf{u}_i, \mathbf{z} \rangle \leq \mathrm{val}(c_{i,\mathbf{v}}) + \langle \mathbf{v}, \mathbf{z} \rangle,\ \text{for } 1 \leq i \leq s, \mathbf{v} \in \mathbb{N}^{n+1}\ \},$$

is proven in the proof of the following:

Proposition (Proposition 2.5.2)

The set $\overline{C_I[\mathbf{w}]}$ is a Γ-rational polyhedron which contains the line $\mathbb{R}(1, 1, \ldots, 1)$ as its largest affine subspace.
If $\mathrm{in}_{\mathbf{w}}(I)$ is not a monomial ideal, then there exists $\mathbf{v} \in \Gamma_{\mathrm{val}}^{n+1}$ such that $\mathrm{in}_{\mathbf{v}}(I)$ is a monomial ideal and $\overline{C_I[\mathbf{w}]}$ is a proper face of $\overline{C_I[\mathbf{v}]}$.

properties of $C_I[\mathbf{w}]$

The *lineality space* V of a polyhedron P is the largest affine subspace contained in P. We have that $\mathbf{x} \in P$, $\mathbf{v} \in V$ implies $\mathbf{x} + \mathbf{v} \in P$.

Denote by $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^{n+1}$. Recall that I is homogeneous. The line $\mathbb{R}\mathbf{1}$ is the lineality space of $\overline{C_I[\mathbf{w}]}$.

Theorem (Theorem 2.5.3)

The polyhedra $\overline{C_I[\mathbf{w}]}$ as $\mathbf{w}$ varies over $\Gamma_{\rm val}^{n+1}$ form a $\Gamma_{\rm val}$-polyhedral complex inside the n-dimensional space $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Lemma (Lemma 2.5.4)

Let I be a homogeneous ideal in $K[x_0, x_1, \ldots, x_n]$. There are only finitely many distinct monomial initial ideals $\mathrm{in}_{\mathbf{w}}(I)$ as $\mathbf{w}$ runs over $\Gamma_{\rm val}^{n+1}$.

the polyhedral complex

Given a tropical polynomial function $F : \mathbb{R}^{n+1} \to \mathbb{R}$, we write Σ_F for the coarsest polyhedral complex such that F is linear on each cell in Σ_F.

The maximal cells of the polyhedral complex Σ_F have the form

$$\sigma = \{ \mathbf{w} \in \mathbb{R}^{n+1} : F(\mathbf{w}) = a + \langle \mathbf{w}, \mathbf{u} \rangle \}$$

where $a \odot \mathbf{x}^{\mathbf{u}}$ runs over monomials of F.

We have $|\Sigma_F| = \mathbb{R}^{n+1}$.

If the coefficients a in $a \odot \mathbf{x}^{\mathbf{u}}$ lie in a subgroup $\Gamma \subset \mathbb{R}$,
then the complex Σ_F is Γ-rational.

A polyhedral complex Σ is Γ-rational if every $P \in \Sigma$ is Γ-rational.

fix an arbitrary homogeneous ideal

By Lemma 2.5.4, there exists $D \in \mathbb{N}$ such that any initial monomial ideal $\mathrm{in}_{\mathbf{w}}(I)$ has generators of degree at most D.

Let $\mathcal{M}_d$ be the set of monomials of degree d in $K[x_0, x_1, \ldots, x_n]$. The coefficients of a basis $\{f_1, f_2, \ldots, f_s\}$ are stored in the matrix A_d. The rows of A_d are indexed by $\mathcal{M}_d$. For $|J| = s$, A_d^J is the s-by-s minor of A_d with column indexed by J. We define the polynomial

$$g := \prod_{d=1}^{D} g_d, \quad \text{where} \quad g_d := \sum_{\substack{J \subseteq \mathcal{M}_d \\ |J| = s}} \det(A_d^J) \prod_{\mathbf{u} \in J} \mathbf{x}^{\mathbf{u}}$$

Theorem (Theorem 2.5.6)

If $\mathbf{w} \in \Gamma^{n+1}$ lies in the interior of a maximal cell $\sigma \in \Sigma_{\mathrm{trop}(g)}$, then $\sigma = C_I[\mathbf{w}]$.

the Gröbner complex

Definition (Definition 2.5.7)

For a homogeneous ideal I in $K[x_0, x_1, \ldots, x_n]$, *the Gröbner complex* $\Sigma(I)$ consists of the polyhedra $\overline{C_I[\mathbf{w}]}$ as $\mathbf{w}$ ranges over $\Gamma_{\mathrm{val}}^{n+1}$.

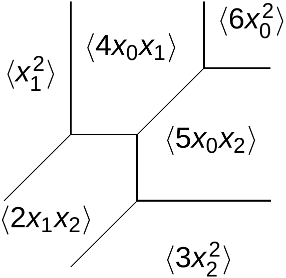
The line $\mathbb{R}\mathbf{1}$ is the lineality space of $\Sigma(I)$.
We identify $\Sigma(I)$ with the quotient complex in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

$\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is called the *tropical projective torus*.

Points in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ can be uniquely represented by vectors of the form $(0, v_1, v_2, \ldots, v_n)$.

an example

$f = t\, x_1^2 + 2x_1x_2 + 3t\, x_2^2 + 4x_0x_1 + 5x_0x_2 + 6t\, x_0^2 \in \mathbb{C}\{\{t\}\}[x_0, x_1, x_2]$,
$I = \langle f \rangle$.



The initial ideal $\mathrm{in}_{\mathbf{w}}(I)$ contains a monomial
if and only if the corresponding cell is full dimensional.

Gröbner Complexes and Tropical Bases

Laurent polynomials

Consider $f \in K[\mathbf{x}^{\pm \mathbf{1}}] = K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Initial forms and initial ideals are defined as before, however ...

For generic **w**:

- the initial form $\mathrm{in}_{\mathbf{w}}(f)$ is just one monomial in $\mathbb{K}[\mathbf{x}^{\pm \mathbf{1}}]$,
- any monomial in $\mathbb{K}[\mathbf{x}^{\pm \mathbf{1}}]$ is a unit, and
- therefore $\mathrm{in}_{\mathbf{w}}(I) = \langle 1 \rangle = \mathbb{K}[\mathbf{x}^{\pm \mathbf{1}}]$.

Tropical geometry is concerned with the study of those **w** for which $\mathrm{in}_{\mathbf{w}}(I)$ is proper in $\mathbb{K}[\mathbf{x}^{\pm \mathbf{1}}]$.

computing with Laurent ideals

Consider a Laurent ideal I in $K[\mathbf{x}^{\pm 1}]$.

The homogenization of $I \subset K[\mathbf{x}^{\pm 1}]$ is
the ideal $I_{\rm proj} \subset K[x_0, x_1, \ldots, x_n]$ of all polynomials

$$x_0^m f\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right), \quad f \in I,$$

where m is the smallest integer that clears the denominator.

To compute ${\rm in}_{\mathbf{w}}(I)$, the weight vectors for the homogeneous ideal $I_{\rm proj}$

- live in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$, and
- we identify $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ with $\mathbb{R}^n$, via $\mathbf{w} \mapsto (0, \mathbf{w})$.

initial ideals of Laurent ideals

Proposition (Proposition 2.6.2)

Let I be an ideal in $K[\mathbf{x}^{\pm 1}]$ and fix $\mathbf{w} \in \Gamma_{\rm val}^n$.
Then $\mathrm{in}_{\mathbf{w}}(I)$ is the image of $\mathrm{in}_{(0,\mathbf{w})}(I_{\rm Proj})$ obtained by $x_0 = 1$.
Every element of $\mathrm{in}_{\mathbf{w}}(I)$ has the form $\mathbf{x}^{\mathbf{u}} g$ where $\mathbf{x}^{\mathbf{u}}$ is a Laurent monomial and $g = f(1, x_1, x_2, \ldots, x_n)$ for some $f \in \mathrm{in}_{(0,\mathbf{w})}(I_{\rm proj})$.

Lemma (Lemma 2.6.3)

Let I be an ideal in $K[\mathbf{x}^{\pm 1}]$ and fix $\mathbf{w} \in \Gamma_{\rm val}^n$.

1. *If $g \in \mathrm{in}_{\mathbf{w}}(I)$, then $g = \mathrm{in}_{\mathbf{w}}(h)$ for some $h \in I$.*
2. *If $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}}(I)$ for some $\mathbf{v} \in \mathbb{Z}^n$, then $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the grading given by $\deg(x_i) = v_i$.*
3. *If $f, g \in K[\mathbf{x}^{\pm 1}]$, then $\mathrm{in}_{\mathbf{w}}(fg) = \mathrm{in}_{\mathbf{w}}(f)\mathrm{in}_{\mathbf{w}}(g)$.*

tropical basis

Let I be an ideal in the Laurent polynomial ring $K[\mathbf{x}^{\pm 1}]$ over a field K with a valuation.

A finite generating set $\mathcal{T}$ of I is a *tropical basis* if for all $\mathbf{w} \in \Gamma_{\rm val}^n$,

$$\mathrm{in}_{\mathbf{w}}(I) \text{ contains a unit}$$

$$\Leftrightarrow \mathrm{in}_{\mathbf{w}}(\mathcal{T}) = \{ \mathrm{in}_{\mathbf{w}}(f) : f \in \mathcal{T} \} \text{ contains a unit.}$$

Theorem (Theorem 2.6.5)

Every ideal I in $K[\mathbf{x}^{\pm 1}]$ has a finite tropical basis.