MATH 425, Linear Algebra II, **Practise Test Final**,
April 30, 2009, Instructor: S. Friedland
Name and e-mail address:

No books or notes. Show all your work. Write solutions in the exam booklet without copying the problems. You can use a result (x) of any part of the problem, to show other part of any problem. **Unjustified** answer yields no credit.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} -2 & -4 \\ 1 & -6 \end{bmatrix}$$

a. Find the characteristic polynomial.
b. Find the eigenvalues and corresponding linear independent eigenvectors.
c. Find the general solution to the system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.

Problem 2. Let $A \in \mathbb{C}^{n\times n}$. Assume that A has only two linearly independent eigenvectors.

1. Show that A has a has at most two distinct eigenvalues.
2. Assume that $n = 3$. Write down all possible Jordan canonical forms of A.
3. Let $n = 3$ and assume that A^2 is diagonable. Suppose that the trace of A is equal to 3. What is the Jordan canonical form of A?

Problem 3. Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

(a) Show that the characteristic and the minimal polynomial of A are equal to $(z-2)^2(z+4)$.
(b) Find the components of A.
(c) Find the formula for A^l for any integer l using the components of A.

Problem 5. Suppose that $A \in \mathbb{C}^{n\times n}$ and the minimal polynomial of A is $(z-\alpha)(z-\beta)^2(z-\gamma)^3$ where α, β, γ are three distinct complex numbers.

(a) Write down the general form of the characteristic polynomial of A.
(b) What is the condition for A to be power stable, i.e. $\lim_{l\to\infty} A^l = 0$.
(c) What is the condition for A to be power convergent, i.e. $\lim_{l\to\infty} A^l = B$.

Problem 6. Let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation on a finite dimensional vector space $\mathbf{V}$ over a field $\mathbb{F}$.
a. Let $\mathbf{U}$ be a nontrivial subspace of $\mathbf{V}$, i.e. $0 < \dim \mathbf{U} < \dim \mathbf{V}$. Define the quotient space $\hat{\mathbf{V}} := \mathbf{V}/\mathbf{U}$ and show that it is a vector space over $\mathbb{F}$ of dimension $\dim \mathbf{V} - \dim \mathbf{U}$.
b. Suppose that $\mathbf{U}$ is T-invariant. Show that T induces a linear transformation $\hat{T} : \hat{\mathbf{V}} \to \hat{\mathbf{V}}$.
c. Let $\psi, \hat{\psi}$ be the minimal polynomials of $T, \hat{T}$ respectively. Show that $\hat{\psi}$ divides ψ.

Problem 7.

1. Let $B = [b_{ij}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that $\langle \mathbf{x}, \mathbf{y}\rangle := \mathbf{y}^\top B\mathbf{x}$ is an inner product on $\mathbb{R}^n$ if and only if B is positive definite.
2. $A \in \mathbb{R}^{m\times n}$. Is $B = A^\top A$ nonnegative definite? **Justify**. Let $A = [a_{ij}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$.
3. Assume that $a_{ij} \in (0, 2]$ for $i, j = 1, \ldots, n$ and $n \geq 2$. Show that $|\det A| \leq 2^n n^{\frac{n}{2}}$. Can equality hold for some matrix A?

Problem 8. Let $B = [b_{ij}]_{i,j=1}^{n} \mathbb{R}^{n\times n}$ be a real symmetric matrix. Denote by $A = [b_{ij}]_{i,j=1}^{n-1}$ the real symmetric matrix obtained from B by deleting the $n - th$ row and column.

1. Show the Cauchy interlacing inequalities

$$\lambda_i(B) \geq \lambda_i(A) \geq \lambda_{i+1}(B), \text{ for } i = 1, ..., n-1.$$

2. Show that inequality $\lambda_1(B) + \lambda_n(B) \leq \lambda_1(A) + b_{nn}$.

Problem 9. Let $A \in \mathbb{R}^{m\times n}$. Recall that the singular value decomposition of A is given as $A = U\Sigma V$, where $U \in \mathbb{R}^{m\times n}, V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and $\Sigma \in \mathbb{R}^{m\times n}$ is a diagonal matrix with the nonnegative diagonal entries $\sigma_1(A) \geq \sigma_2(A) \geq \ldots$, which are called the singular values of A. Show

1. $\sigma_1(A) = \max_{\mathbf{0}\neq\mathbf{x}\in\mathbb{R}^n} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$. Here $\|\mathbf{x}\| = \sqrt{\mathbf{x}^\top\mathbf{x}}$. **Hint**: $\|A\mathbf{x}\|^2 = \mathbf{x}^\top A^\top A\mathbf{x}$.
2. $\sigma_1(A + B) \leq \sigma_1(A) + \sigma_1(B)$ for any $A, B \in \mathbb{R}^{m\times n}$. **Hint**: Use the triangle inequality $\|\mathbf{y} + \mathbf{z}\| \leq \|\mathbf{y}\| + \|\mathbf{z}\|$.
3. Assume that $A \in \mathbb{R}^{n\times n}$ is invertible. Show that $\sigma_1(A^{-1}) = (\sigma_n(A))^{-1}$.