

No books or notes. Show all your work. Write solutions in the exam booklet without copying the problems. **Unjustified** answer yields no credit.

Problem 1. Let $A \in \mathbb{F}^{n \times n}$ matrix.

- Define the characteristic and the minimal polynomial of A .
- State the Cayley-Hamilton theorem.
- Give a proof of Cayley-Hamilton theorem for a diagonalizable matrix A .
- Show that the minimal polynomial of A divides its characteristic polynomial.
- Assume that λ is an eigenvalue of A . Show that λ is a root of the characteristic and minimal polynomial of A .

Problem 2. Let $A = \begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$

- Find the characteristic polynomial of A .
- Show that A is diagonalizable over \mathbb{R} .
- Find $P \in \text{GL}(3, \mathbb{R})$ and a diagonal D such that $A = PDP^{-1}$.
- Find the minimal polynomial of A .

Problem 3. Suppose that the characteristic and the minimal polynomial of A is $(z - 4)^4$.

- Write down the Jordan canonical form of A .
- What is the Jordan canonical form of A^2 ?
- Show that the matrix equation $X^2 = A$ is always solvable. Is a solution unique?
- Show that the equation $Y^2 = A - 4I$ is not solvable.

Problem 4. Assume that the characteristic polynomial of A is $(z - 1)^3(z + 2)^2$.

- What is the maximal possible degree of the minimal polynomial of A . Write down all possible Jordan canonical forms of A for this minimal polynomial.
- What is the minimal possible degree of the minimal polynomial of A . Write down all possible Jordan canonical forms of A for this minimal polynomial.
- Is it possible that A has exactly 3 independent eigenvectors? If yes, write down all possible Jordan canonical forms of A .

Problem 5. Let $T : \mathbf{V} \rightarrow \mathbf{V}$ be a linear transformation on a finite dimensional vector space \mathbf{V} over a field \mathbb{F} .

- What does it mean that a subspace \mathbf{U} is a nontrivial T -invariant subspace?
- What is a cyclic invariant subspace generated by $\mathbf{u} \neq \mathbf{0}$.
- Let ψ be the minimal polynomial of T assume that $\psi = \psi_1\psi_2$, where ψ_1, ψ_2 are monic polynomials of degree one at least which are co-prime. Show that $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$, where $\mathbf{V}_1, \mathbf{V}_2$ are two nontrivial invariant subspaces of T such that the minimal polynomial of the restriction of T to \mathbf{V}_i is ψ_i for $i = 1, 2$.
- Let $\mathbf{0} \neq \mathbf{u}, \mathbf{w} \in \mathbf{V}$. Denote by $\psi_{\mathbf{u}}, \psi_{\mathbf{w}}$ the minimal polynomial of T restricted to the cyclic subspaces generated by \mathbf{u}, \mathbf{w} respectively. Assume that $(\psi_{\mathbf{u}}, \psi_{\mathbf{w}}) = 1$. Show that $\psi_{\mathbf{u}+\mathbf{w}} = \psi_{\mathbf{u}}\psi_{\mathbf{w}}$.