Generalized rank-constrained matrix approximations

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Abstract

In this paper we give an explicit solution to the rank constrained matrix approximation in Frobenius norm, which is a generalization of the classical approximation of an $m \times n$ matrix A by a matrix of rank k at most.

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1 Introduction

Let $\mathbb{C}^{m\times n}$ be set of $m\times n$ complex valued matrices, and denote by $\mathcal{R}(m,n,k) \subseteq \mathbb{C}^{m\times n}$ the variety of all $m \times n$ matrices of rank k at most. Fix $A = [a_{ij}]_{i,j=1}^{m,n} \in \mathbb{C}^{m\times n}$. Then $A^* \in \mathbb{C}^{n\times m}$ is the conjugate transpose of A, and $||A||_F := \sqrt{\sum_{i,j=1}^{m,n} |a_{ij}|^2}$ is the Frobenius norm of A. Recall that the *singular value decomposition* of A, abbreviated here as *SVD*, is given by $A = U_A \Sigma_A V_A^*$, where $U_A \in \mathbb{C}^{m\times m}, V_A \in \mathbb{C}^{n\times n}$ are unitary matrices, $\Sigma_A := \mathrm{diag}(\sigma_1(A), \ldots, \sigma_{\min(m,n)}(A)) \in \mathbb{C}^{m\times n}$ is a generalized diagonal matrix, with the singular values $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq 0$ on the main diagonal. The number of positive singular values of A is r, which is equal to the rank of A, denoted by rank A. Let $U_A = [\mathbf{u}_1\ \mathbf{u}_2\ \ldots \mathbf{u}_m], V_A = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots \mathbf{v}_n]$ be the representations of U, V in terms of their m, n columns respectively. Then $\mathbf{u}_i$ and $\mathbf{v}_i$ are called the *left* and the *right* singular vectors of A, respectively, that correspond to the singular value $\sigma_i(A)$. Let

$$P_{A,L} := \sum_{i=1}^{\mathrm{rank}\,A} \mathbf{u}_i \mathbf{u}_i^* \in \mathbb{C}^{m\times m}, \quad P_{A,R} := \sum_{i=1}^{\mathrm{rank}\,A} \mathbf{v}_i \mathbf{v}_i^* \in \mathbb{C}^{n\times n}, \tag{1.1}$$

be the orthogonal projections on the range of A and A^* respectively. Denote by

$$A_k := \sum_{i=1}^{k} \sigma_i(A) \mathbf{u}_i \mathbf{v}_i^* \in \mathbb{C}^{m\times n}$$

for $k = 1, \ldots, \mathrm{rank}\,A$. For $k > \mathrm{rank}\,A$ we define $A_k := A$ $(= A_{\mathrm{rank}\,A})$. For $1 \leq k < \mathrm{rank}\,A$, the matrix A_k is uniquely defined if and only if $\sigma_k(A) > \sigma_{k+1}(A)$.

The enormous application of SVD decomposition of A in pure and applied mathematics, is derived from the following approximation property:

$$\min_{X\in\mathcal{R}(m,n,k)} ||A-X||_F = ||A-A_k||_F, \quad k=1,\ldots \tag{1.2}$$

The latter is known as the Eckart-Young theorem [2]. We note that the work [2] implied a number of extensions. We cite [4, 5, 7, 8] as some recent references. Another application of SVD is a formula for the Moore-Penrose inverse $A^\dagger := V_A\Sigma_A^\dagger U_A^* \in \mathbb{C}^{n\times m}$ of A, where
$\Sigma_A^\dagger := \mathrm{diag}(\frac{1}{\sigma_1(A)},\ldots,\frac{1}{\sigma_{\mathrm{rank}\ A}(A)},0,\ldots,0) \in \mathbb{C}^{n\times m}$. See for example [1].

2 Main Result

Below, we provide generalizations of the classical minimal problem given in (1.2).

Theorem 2.1 *Let matrices $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{m\times p}$ and $C\in\mathbb{C}^{q\times n}$ be given. Then*

$$X = B^\dagger (P_{B,L}AP_{C,R})_k C^\dagger \tag{2.1}$$

is a solution to the minimal problem

$$\min_{X\in\mathcal{R}(p,q,k)} ||A-BXC||_F, \tag{2.2}$$

having the minimal $||X||_F$. This solution is unique if and only if either

$$k \ge \mathrm{rank}\ P_{B,L}AP_{C,R}$$

or

$$1\le k < \mathrm{rank}\ P_{B,L}AP_{C,R} \quad \textit{and} \quad \sigma_k(P_{B,L}AP_{C,R}) > \sigma_{k+1}(P_{B,L}AP_{C,R}).$$

Proof of Theorem 2.1 Recall that the Frobenius norm is invariant under the multiplication from the left and the right by the corresponding unitary matrices. Hence $||A-BXC||_F = ||\widetilde{A}-\Sigma_B\widetilde{X}\Sigma_C||_F$, where $\widetilde{A} := U_B^*AV_C$ and $\widetilde{X} := V_B^*XU_C$. Clearly, X and $\widetilde{X}$ have the same rank and the same Frobenius norm. Thus, it is enough to consider the minimal problem $\min\limits_{\widetilde{X}\in\mathcal{R}(p,q,k)} ||\widetilde{A}-\Sigma_B\widetilde{X}\Sigma_C||_F$.

Let $s=\mathrm{rank}\ B$ and $t=\mathrm{rank}\ C$. Clearly if B or C is a zero matrix, then $X=\mathbf{0}$ is the solution to the minimal problem (2.2). In this case either $P_{B,L}$ or $P_{C,R}$ are zero matrices, and the theorem holds trivially in this case.

Let us consider the case $1\le s, 1\le t$. Define $B_1 := \mathrm{diag}(\sigma_1(B),\ldots,\sigma_s(B)) \in \mathbb{C}^{s\times s}, C_1 := \mathrm{diag}(\sigma_1(C),\ldots,\sigma_t(C)) \in \mathbb{C}^{t\times t}$. Partition $\widetilde{A}$ and $\widetilde{X}$ into four block matrices A_{ij} and X_{ij} with $i,j=1,2$ so that $\widetilde{A}=[A_{ij}]_{i,j=1}^2$ and $\widetilde{X}=[X_{ij}]_{i,j=1}^2$, where $A_{11}, X_{11}\in\mathbb{C}^{s\times t}$. (For certain values of s and t, we may have to partition $\widetilde{A}$ or $\widetilde{X}$ to less than four block matrices.) Next, observe that $Z := \Sigma_B\tilde{X}\Sigma_C = [Z_{ij}]_{i,j=1}^2$, where $Z_{11}=B_1X_{11}C_1$ and all other blocks Z_{ij} are zero matrices. Since B_1 and C_1 are invertible we deduce

$$\mathrm{rank}\ Z = \mathrm{rank}\ Z_{11} = \mathrm{rank}\ X_{11} \le \mathrm{rank}\ \widetilde{X} \le k.$$

The approximation property of $(A_{11})_k$ yields the inequality $||A_{11} - Z_{11}||_F \geq ||A_{11} - (A_{11})_k||_F$ for any Z_{11} of rank k at most. Hence for any Z of the above form,

$$||\widetilde{A} - Z||_F^2 = ||A_{11} - Z_{11}||_F^2 + \sum_{2<i+j\leq 4} ||A_{ij}||_F^2 \geq ||A_{11} - (A_{11})_k||_F^2 + \sum_{2<i+j\leq 4} ||A_{ij}||_F^2.$$

Thus $\widehat{X} = [X_{ij}]_{i,j=1}^2$, where $X_{11} = B_1^{-1}(A_{11})_k C_1^{-1}$ and $X_{ij} = \mathbf{0}$ for all $(i,j) \neq (1,1)$ is a solution to the problem $\min\limits_{\widetilde{X}\in\mathcal{R}(p,q,k)} ||\widetilde{A} - \Sigma_B \widetilde{X} \Sigma_C||_F$ with the minimal Frobenius form. This solution is unique if and only if the solution $Z_{11} = (A_{11})_k$ is the unique solution to the problem $\min\limits_{Z_{11}\in\mathcal{R}(s,t,k)} ||A_{11} - Z_{11}||_F$. This happens if either $k \geq \operatorname{rank} A_{11}$ or $1 \leq k < \operatorname{rank} A_{11}$ and $\sigma_k(A_{11}) > \sigma_{k+1}(A_{11})$. A straightforward calculation shows that $\widehat{X} = \Sigma_B^\dagger (P_{\Sigma_B,L} \widetilde{A} P_{\Sigma_C,R})_k \Sigma_C^\dagger$. Thus, a solution of (2.2) with the minimal Frobenius norm is given by

$$\begin{aligned} X &= B^\dagger U_B (P_{\Sigma_B,L} U_B^* A V_C P_{\Sigma_C,R})_k V_C^* C^\dagger \\ &= B^\dagger U_B (U_B^* P_{B,L} A P_{C,R} V_C)_k V_C^* C^\dagger \\ &= B^\dagger (P_{B,L} A P_{C,R})_k C^\dagger. \end{aligned}$$

This solution is unique if and only if either $k \geq \operatorname{rank} P_{B,L}AP_{C,R}$, or $1 \leq k < \operatorname{rank} P_{B,L}AP_{C,R}$ and $\sigma_k(P_{B,L}AP_{C,R}) > \sigma_{k+1}(P_{B,L}AP_{C,R})$. □

3 Examples

First observe that the classical approximation problem given by (1.2) is equivalent to the case $m = p, n = q, B = I_m, C = I_n$. (Here, I_m is the $m \times m$ identity matrix.) Clearly $P_{I_m,L} = I_m,\ P_{I_n,R} = I_n,\ I_m^\dagger = I_m,\ I_n^\dagger = I_n$. In this case we obtain the classical solution $B^\dagger (P_{B,L}AP_{C,R})_k C^\dagger = A_k$.

Second, if $p = m$, $q = n$ and B, C are non-singular, then $\operatorname{rank}(BXC) = \operatorname{rank} X$. In this case, $P_{B,L} = I_m$ and $P_{C,R} = I_n$, and the solution to (2.2) is given by $X = B^{-1} A_k C^{-1}$.

Next, a particular case of the problem (2.2) occurs in study of a random vector estimation (see, for example, [9, 6]) as follows. Let (Ω, Σ, μ) be a probability space, where Ω is the set of outcomes, Σ a σ–field of measurable subsets $\Delta \subset \Omega$ and $\mu : \Sigma \mapsto [0,1]$ an associated probability measure on Σ with $\mu(\Omega) = 1$. Suppose that $\mathbf{x} \in L^2(\Omega, \mathbb{R}^m)$ and $\mathbf{y} \in L^2(\Omega, \mathbb{R}^n)$ are random vectors such that $\mathbf{x} = (x_1, \ldots, x_m)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$ with $x_i, y_j \in L^2(\Omega, \mathbb{R})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, respectively. Let $E_{xy} = [e_{ij,xy}] \in \mathbb{R}^{m\times n}$, $E_{yy} = [e_{jk,yy}] \in \mathbb{R}^{n\times n}$ be correlation matrices with entries

$$e_{ij,xy} = \int_\Omega x_i(\omega) y_j(\omega) d\mu(\omega), \quad e_{jk,yy} = \int_\Omega y_j(\omega) y_k(\omega) d\mu(\omega),$$
$$i = 1, \ldots, m, \quad j,k = 1, \ldots, n, \quad \omega \in \Omega.$$

The problems considered in [9, 6] are reduced to finding a solution to the problem (2.2) with $A = E_{xy} E_{yy}^{1/2\dagger}$, $B = I_n$ and $C = E_{yy}^{1/2}$ where we write $E_{yy}^{1/2\dagger} = (E_{yy}^{1/2})^\dagger$.

Let the SVD of $E_{yy}^{1/2}$ be given by $E_{yy}^{1/2} = V_n \Sigma V_n^*$ and let rank $E_{yy}^{1/2} = r$. Here, $V_n = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ with $\mathbf{v}_i$ the i-th column of V_n. By Theorem 2.1, the solution to this particular case of the problem (2.2) having the minimal Frobenius norm is given by $X = (E_{xy} E_{yy}^{1/2\dagger} V_r V_r^*)_k E_{yy}^{1/2\dagger}$, where $E_{yy}^{1/2\dagger} V_r V_r^* = E_{yy}^{1/2\dagger}$. Therefore, $X = (E_{xy} E_{yy}^{1/2\dagger})_k E_{yy}^{1/2\dagger}$. The conditions for the uniqueness follow directly from Theorem 2.1.

Finally, a special case of the minimal problem (2.2), where X is a rank one matrix and C the identity matrix, was considered by Michael Elad [3] in the context of image processing.

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