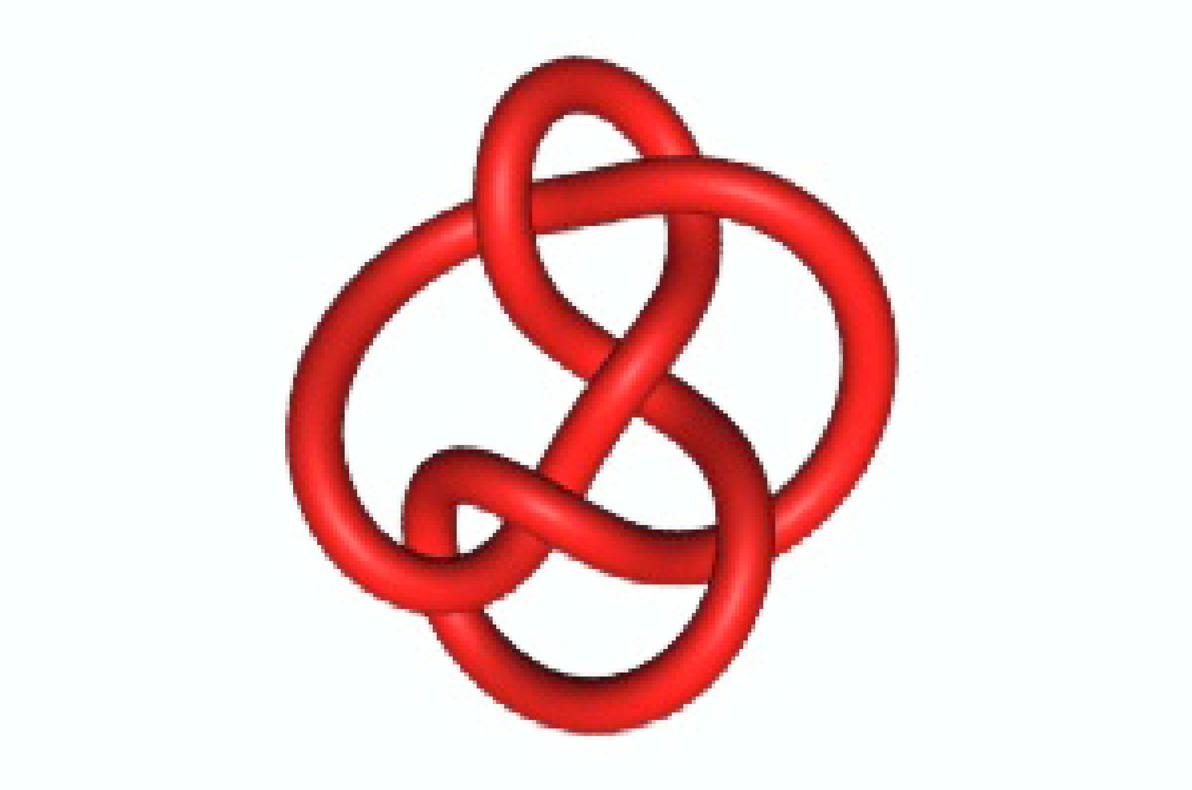
The A-polynomial and the FFT

Marc Culler

anatomy of an A-polynomial

anatomy of an A-polynomial

Start with a (framed) knot complement $\mathcal{M}$, for example 6_3.



(Thanks, KnotPlot.)

anatomy of an A-polynomial

Its A-polynomial is a 2-variable polynomial. ($\widetilde{M} = M^2$.)

$$
\begin{aligned}
A_{6_3}(M, L) = {} & L^3\widetilde{M}^{14} - 5\,L^3\widetilde{M}^{13} + L^2\widetilde{M}^{13} + L^4\widetilde{M}^{13} - 4\,L^4\widetilde{M}^{12} \\
& + 3\,L^3\widetilde{M}^{12} - 4\,L^2\widetilde{M}^{12} + 4\,L^2\widetilde{M}^{11} + 4\,L^4\widetilde{M}^{11} + 9\,L^3\widetilde{M}^{11} \\
& - 2\,L^3\widetilde{M}^{10} + 2\,L^2\widetilde{M}^{10} + 2\,L^5\widetilde{M}^{10} + 2\,L^4\widetilde{M}^{10} + 2\,L\widetilde{M}^{10} \\
& - 5\,L^5\widetilde{M}^{9} - 21\,L^3\widetilde{M}^{9} - 6\,L^4\widetilde{M}^{9} - 5\,L\widetilde{M}^{9} - 6\,L^2\widetilde{M}^{9} + L^5\widetilde{M}^{8} \\
& + 2\,L^4\widetilde{M}^{8} + 8\,L^3\widetilde{M}^{8} + 2\,L^2\widetilde{M}^{8} + L\widetilde{M}^{8} + 34\,L^3\widetilde{M}^{7} + \widetilde{M}^{7} \\
& + L^6\widetilde{M}^{7} + 17\,L^2\widetilde{M}^{7} + 10\,L\widetilde{M}^{7} + 17\,L^4\widetilde{M}^{7} + 10\,L^5\widetilde{M}^{7} + L^5\widetilde{M}^{6} \\
& + 8\,L^3\widetilde{M}^{6} + 2\,L^2\widetilde{M}^{6} + L\widetilde{M}^{6} + 2\,L^4\widetilde{M}^{6} - 21\,L^3\widetilde{M}^{5} - 5\,L\widetilde{M}^{5} \\
& - 6\,L^2\widetilde{M}^{5} - 6\,L^4\widetilde{M}^{5} - 5\,L^5\widetilde{M}^{5} + 2\,L^4\widetilde{M}^{4} + 2\,L^2\widetilde{M}^{4} + 2\,L^5\widetilde{M}^{4} \\
& + 2\,L\widetilde{M}^{4} - 2\,L^3\widetilde{M}^{4} + 4\,L^4\widetilde{M}^{3} + 4\,L^2\widetilde{M}^{3} + 9\,L^3\widetilde{M}^{3} - 4\,L^4\widetilde{M}^{2} \\
& - 4\,L^2\widetilde{M}^{2} + 3\,L^3\widetilde{M}^{2} + L^4\widetilde{M} - 5\,L^3\widetilde{M} + L^2\widetilde{M} + L^3
\end{aligned}
$$

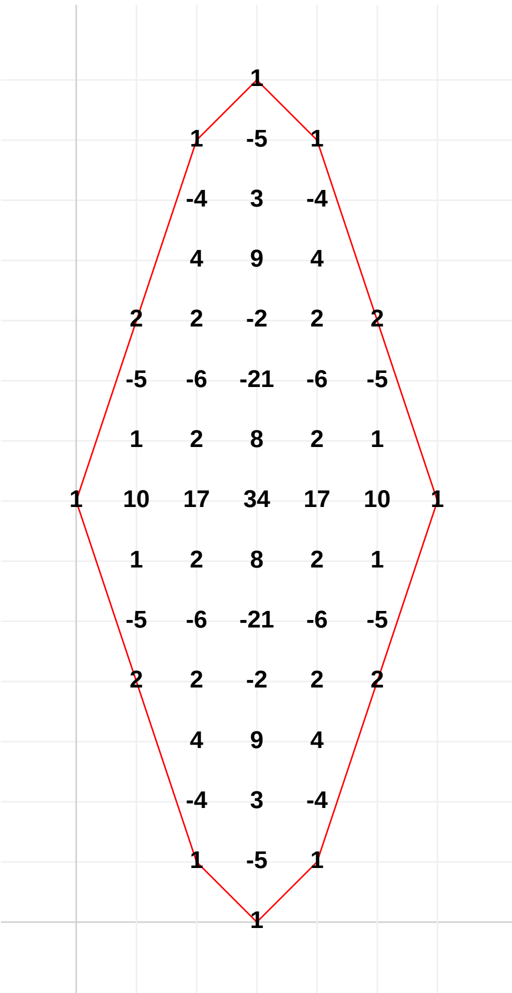
anatomy of an A-polynomial

It helps to plot the coefficient matrix on $\mathbb{Z} \times 2\mathbb{Z}$:

			1			
		1	-5	1		
		-4	3	-4		
		4	9	4		
	2	2	-2	2	2	
	-5	-6	-21	-6	-5	
	1	2	8	2	1	
1	10	17	34	17	10	1
	1	2	8	2	1	
	-5	-6	-21	-6	-5	
	2	2	-2	2	2	
		4	9	4		
		-4	3	-4		
		1	-5	1		
			1			

anatomy of an A-polynomial

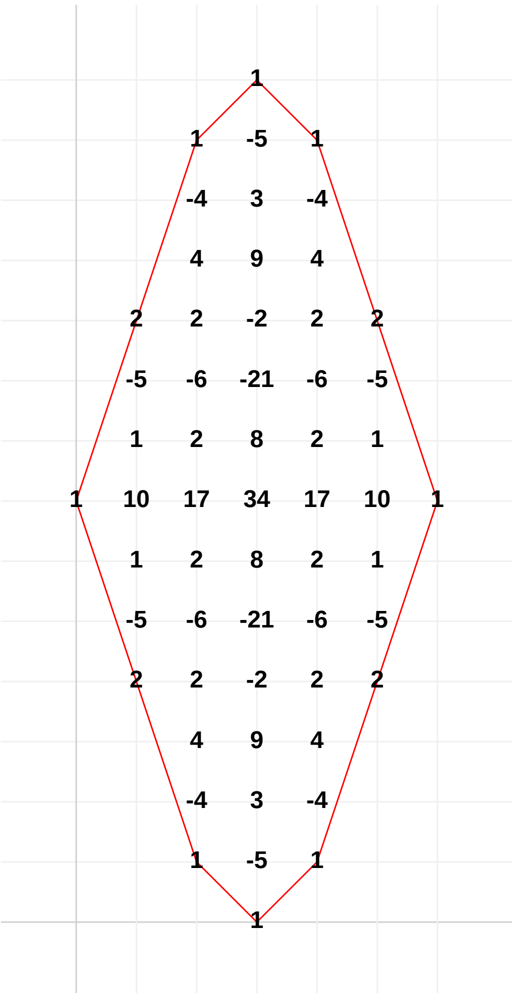
It helps to plot the coefficient matrix on $\mathbb{Z} \times 2\mathbb{Z}$:



- The *Newton polygon* is the convex hull of the non-zero coefficients.

anatomy of an A-polynomial

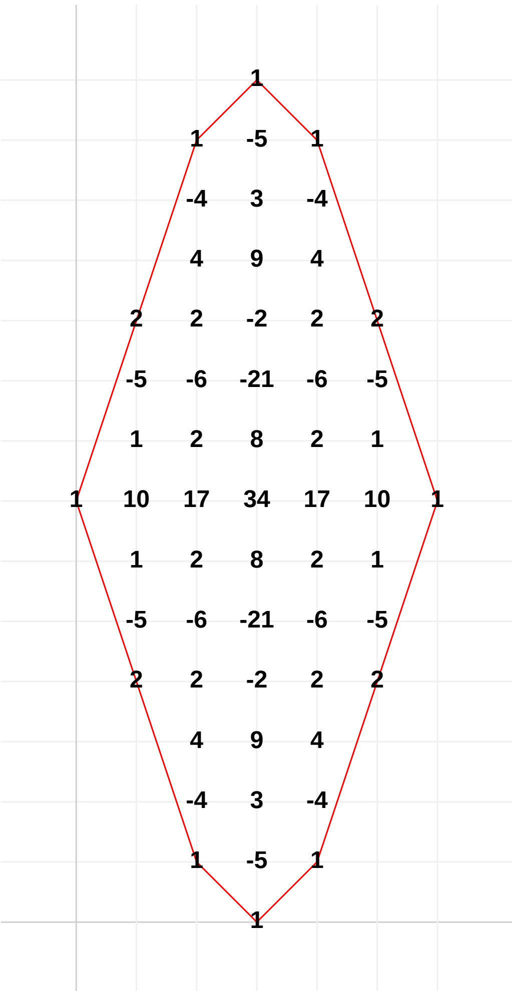
It helps to plot the coefficient matrix on $\mathbb{Z} \times 2\mathbb{Z}$:



- The *Newton polygon* is the convex hull of the non-zero coefficients.
- The slopes of the sides are boundary slopes of essential surfaces. (Here $\pm 6/1$ and $\pm 2/1$.)

anatomy of an A-polynomial

It helps to plot the coefficient matrix on $\mathbb{Z} \times 2\mathbb{Z}$:



- The *Newton polygon* is the convex hull of the non-zero coefficients.
- The slopes of the sides are boundary slopes of essential surfaces. (Here $\pm 6/1$ and $\pm 2/1$.)
- The "edge polynomials" are monic (Cooper-Long) with cyclotomic irreducible factors, and the orders of the roots are related to the number of boundary components of the surface.

about boundary slopes

- A *framing* (μ, λ) of a knot manifold M is a basis for $H_1(\partial M; \mathbb{Z})$. Every simple closed curve on ∂M is homologous to $p\mu + q\lambda$. Its *slope* $p/q \in \mathbb{Q} \cup \{1/0\}$ does not depend on an orientation.

about boundary slopes

- A *framing* (μ, λ) of a knot manifold M is a basis for $H_1(\partial M; \mathbb{Z})$. Every simple closed curve on ∂M is homologous to $p\mu + q\lambda$. Its *slope* $p/q \in \mathbb{Q} \cup \{1/0\}$ does not depend on an orientation.
- A properly embedded bounded surface S in M is *essential* if $\pi_1(S)$ injects in $\pi_1(M)$ and S is not boundary-parallel. All of the boundary curves of an essential surface have the same slope.

about boundary slopes

- A *framing* (μ, λ) of a knot manifold M is a basis for $H_1(\partial M; \mathbb{Z})$. Every simple closed curve on ∂M is homologous to $p\mu + q\lambda$. Its *slope* $p/q \in \mathbb{Q} \cup \{1/0\}$ does not depend on an orientation.
- A properly embedded bounded surface S in M is *essential* if $\pi_1(S)$ injects in $\pi_1(M)$ and S is not boundary-parallel. All of the boundary curves of an essential surface have the same slope.
- Only finitely many slopes arise as boundary slopes of essential surfaces (Hatcher). The A-polynomial detects many, but not all. (E.g. the boundary of the spanning surface of a fibered knot is never detected.)

definition of the A-polynomial

- For $\mathcal{M}$ compact, $R(\mathcal{M}) = \mathrm{Hom}(\pi_1(\mathcal{M}), SL_2(\mathbb{C}))$ is an affine algebraic subset of $SL_2(\mathbb{C})^n$.

definition of the A-polynomial

- For $\mathcal{M}$ compact, $R(\mathcal{M}) = \mathrm{Hom}(\pi_1(\mathcal{M}), SL_2(\mathbb{C}))$ is an affine algebraic subset of $SL_2(\mathbb{C})^n$.
- The *character variety* $X(\mathcal{M}) = R(\mathcal{M})/\sim$ is the algebro-geometric quotient under the conjugation action of $SL_2(\mathbb{C})$.

definition of the A-polynomial

- For $\mathcal{M}$ compact, $R(\mathcal{M}) = \mathrm{Hom}(\pi_1(\mathcal{M}), SL_2(\mathbb{C}))$ is an affine algebraic subset of $SL_2(\mathbb{C})^n$.
- The *character variety* $X(\mathcal{M}) = R(\mathcal{M})/\sim$ is the algebro-geometric quotient under the conjugation action of $SL_2(\mathbb{C})$.
- For a knot manifold $\mathcal{M}$, let $\Delta \subset R(\partial\mathcal{M})$ be the variety of diagonal representations of $\pi_1(\partial\mathcal{M}) = \mathbb{Z}^2$. We have

$$
\begin{tikzcd}
\mathbb{C}^* \times \mathbb{C}^* \cong \Delta \arrow[r, hook] \arrow[dr, "t"'] & R(\partial\mathcal{M}) \arrow[d] & R(\mathcal{M}) \arrow[l] \arrow[d] \\
 & X(\partial\mathcal{M}) & X(\mathcal{M}) \arrow[l, "i_*"]
\end{tikzcd}
$$

definition of the A-polynomial

- For $\mathcal{M}$ compact, $R(\mathcal{M}) = \mathrm{Hom}(\pi_1(\mathcal{M}), SL_2(\mathbb{C}))$ is an affine algebraic subset of $SL_2(\mathbb{C})^n$.
- The *character variety* $X(\mathcal{M}) = R(\mathcal{M})/\sim$ is the algebro-geometric quotient under the conjugation action of $SL_2(\mathbb{C})$.
- For a knot manifold $\mathcal{M}$, let $\Delta \subset R(\partial\mathcal{M})$ be the variety of diagonal representations of $\pi_1(\partial\mathcal{M}) = \mathbb{Z}^2$. We have

$$\begin{tikzcd} \mathbb{C}^* \times \mathbb{C}^* \cong \Delta \arrow[r, hook] \arrow[dr, "t"'] & R(\partial\mathcal{M}) \arrow[d] & R(\mathcal{M}) \arrow[l] \arrow[d] \\ & X(\partial\mathcal{M}) & X(\mathcal{M}) \arrow[l, "i_*"] \end{tikzcd}$$

- The closure of the union of 1-dimensional components of $t^{-1}(i_*(X(\mathcal{M}))$ is a plane algebraic curve $C(\mathcal{M})$.

definition of the A-polynomial

Note: For an oriented hyperbolic knot manifold $\mathcal{M}$, the irreducible component $X_0(\mathcal{M})$ of $X(\mathcal{M})$ passing through the (smooth) point that corresponds to the discrete faithful representation is 1-dimensional.

definition of the A-polynomial

Note: For an oriented hyperbolic knot manifold $\mathcal{M}$, the irreducible component $X_0(\mathcal{M})$ of $X(\mathcal{M})$ passing through the (smooth) point that corresponds to the discrete faithful representation is 1-dimensional.

The A-polynomial is a suitably chosen defining equation of the plane curve $C(\mathcal{M})$. It is well-defined up to scalar multiplication and multiplicities of irreducible factors. Suitably normalized, it has integer coefficients. For a knot in a homology sphere (with the standard framing) it is a polynomial in M^2 and L, where M and L represent eigenvalues of the images of the meridian and longitude under an $SL_2(\mathbb{C})$ representation.

definition of the A-polynomial

Note: For an oriented hyperbolic knot manifold $\mathcal{M}$, the irreducible component $X_0(\mathcal{M})$ of $X(\mathcal{M})$ passing through the (smooth) point that corresponds to the discrete faithful representation is 1-dimensional.

The A-polynomial is a suitably chosen defining equation of the plane curve $C(\mathcal{M})$. It is well-defined up to scalar multiplication and multiplicities of irreducible factors. Suitably normalized, it has integer coefficients. For a knot in a homology sphere (with the standard framing) it is a polynomial in M^2 and L, where M and L represent eigenvalues of the images of the meridian and longitude under an $SL_2(\mathbb{C})$ representation.

For computational reasons we replace $X(\mathcal{M})$ with a closely related "gluing variety" G associated to an ideal triangulation. The defining equations of G are computed by SnapPea. The curve G parametrizes $PSL(2,\mathbb{C})$ representations. By taking square roots carefully, G can be lifted to give a curve in $X(\mathcal{M})$.

a problem in computational algebraic geometry

a problem in computational algebraic geometry

Given:

- defining equations over $\mathbb{Z}$ for an affine algebraic curve C in $\mathbb{C}^n$; and
- two rational functions L and M, defined over $\mathbb{Z}$.

Compute:

- A defining equation $A(M, L) \in \mathbb{Z}[M, L]$ for the closure of the image of the projection $\pi : C \to \mathbb{C}^2$ given by $\pi(x) = (L(x), M(x))$.

a problem in computational algebraic geometry

Given:

- defining equations over $\mathbb{Z}$ for an affine algebraic curve C in $\mathbb{C}^n$; and
- two rational functions L and M, defined over $\mathbb{Z}$.

Compute:

- A defining equation $A(M, L) \in \mathbb{Z}[M, L]$ for the closure of the image of the projection $\pi : C \to \mathbb{C}^2$ given by $\pi(x) = (L(x), M(x))$.

The classical solution comes from elimination theory. Augment the system of defining equations for C by adding the equations that define L and M. Eliminate all variables except for L amd M. This can be done with resultants or Gröbner bases.

a problem in computational algebraic geometry

Given:

- defining equations over $\mathbb{Z}$ for an affine algebraic curve C in $\mathbb{C}^n$; and
- two rational functions L and M, defined over $\mathbb{Z}$.

Compute:

- A defining equation $A(M, L) \in \mathbb{Z}[M, L]$ for the closure of the image of the projection $\pi : C \to \mathbb{C}^2$ given by $\pi(x) = (L(x), M(x))$.

The classical solution comes from elimination theory. Augment the system of defining equations for C by adding the equations that define L and M. Eliminate all variables except for L amd M. This can be done with resultants or Gröbner bases.

We use interpolation instead.

special features

Two special features of our situation can be used to advantage by the interpolation method.

special features

Two special features of our situation can be used to advantage by the interpolation method.

1. It is easy to use Newton's method to find lots of solutions to polynomial systems. By solving and projecting we can easily sample our plane curve at many points. We get SnapPea to do this job for us.

special features

Two special features of our situation can be used to advantage by the interpolation method.

1. It is easy to use Newton's method to find lots of solutions to polynomial systems. By solving and projecting we can easily sample our plane curve at many points. We get SnapPea to do this job for us.

2. Because the coefficients of the A-polynomial are integers, and because we know that the coefficients at the vertices of the Newton polygon are ± 1, the normalization is relatively easy. Also, it is relatively easy to verify that a polynomial relation holds by symbolic computation.

special features

Two special features of our situation can be used to advantage by the interpolation method.

1. It is easy to use Newton's method to find lots of solutions to polynomial systems. By solving and projecting we can easily sample our plane curve at many points. We get SnapPea to do this job for us.

2. Because the coefficients of the A-polynomial are integers, and because we know that the coefficients at the vertices of the Newton polygon are ± 1, the normalization is relatively easy. Also, it is relatively easy to verify that a polynomial relation holds by symbolic computation.

Nonetheless, the integers involved can be quite large, and care is needed to make the interpolation computationally stable and fast. This is where the FFT comes in.

about the FFT

Fix a dimension N and set $\zeta_N = e^{2\pi i/N}$. The FFT is the algebra isomorphism between $\mathcal{L}_N = \mathbb{C}[x]/(x^N - 1)$ and the direct product $\mathbb{C}^N$ (component-wise multiplication), given by

$$FFT([f]) = (f(1), f(\zeta_N^{-1}), f(\zeta_N^{-2}), \dots, f(\zeta_N^{1-N})).$$

about the FFT

Fix a dimension N and set $\zeta_N = e^{2\pi i/N}$. The FFT is the algebra isomorphism between $\mathcal{L}_N = \mathbb{C}[x]/(x^N - 1)$ and the direct product $\mathbb{C}^N$ (component-wise multiplication), given by

$$FFT([f]) = (f(1), f(\zeta_N^{-1}), f(\zeta_N^{-2}), \dots, f(\zeta_N^{1-N})).$$

Think of the algebra $\mathcal{L}_N$ as a computer model for the ring of Laurent polynomials, or series. So take the standard basis for $\mathcal{L}_N$ to be

$$(1, x, x^2, \dots, x^{[\frac{N}{2}]}, x^{-1}, x^{-2}, \dots, x^{-[\frac{N-1}{2}]}).$$

about the FFT

Fix a dimension N and set $\zeta_N = e^{2\pi i/N}$. The FFT is the algebra isomorphism between $\mathcal{L}_N = \mathbb{C}[x]/(x^N - 1)$ and the direct product $\mathbb{C}^N$ (component-wise multiplication), given by

$$FFT([f]) = (f(1), f(\zeta_N^{-1}), f(\zeta_N^{-2}), \dots, f(\zeta_N^{1-N})).$$

Think of the algebra $\mathcal{L}_N$ as a computer model for the ring of Laurent polynomials, or series. So take the standard basis for $\mathcal{L}_N$ to be

$$(1, x, x^2, \dots, x^{[\frac{N}{2}]}, x^{-1}, x^{-2}, \dots, x^{-[\frac{N-1}{2}]}).$$

Computing the FFT in the standard bases means multiplying by a unitary matrix with all entries of modulus 1. The Cooley-Tukey algorithm does this in essentially linear time in N, and it is numerically stable.

about the FFT

The key point for us is:

If $f(x)$ is a polynomial of degree less than $N/2$, sampled at the N^{th} roots of 1, then the coefficients of f can be computed as

$$FFT^{-1}((f(1), f(\zeta_N^{-1}), f(\zeta_N^{-2}), \ldots, f(\zeta_N^{1-N})).$$

strategy

Write $A(M, L) = \sum_{n=0}^{D} a_n(M) L^n$.

strategy

Write $A(M, L) = \sum_{n=0}^{D} a_n(M) L^n$.

Make it monic in L: $\hat{A}(M, L) = L^D + \sum_{n=0}^{D-1} \hat{a}_n(M) L^n$ where $\hat{a}_n(M) = a_n(M)/a_D(M)$. NOTE: often $a_D = M^k$, e.g. whenever ∞ is not a detected boundary slope. In these cases the $\hat{a}_n$ are Laurent polynomials.

strategy

Write $A(M,L) = \sum_{n=0}^{D} a_n(M)L^n$.

Make it monic in L: $\hat{A}(M,L) = L^D + \sum_{n=0}^{D-1} \hat{a}_n(M)L^n$ where $\hat{a}_n(M) = a_n(M)/a_D(M)$. NOTE: often $a_D = M^k$, e.g. whenever ∞ is not a detected boundary slope. In these cases the $\hat{a}_n$ are Laurent polynomials.

Treat M as a parameter, so $\hat{A}(z,L) = \prod_{i=1}^{D}(L - \lambda_i(z))$, where $\{\lambda_1(z), \ldots, \lambda_D(z)\}$ are the roots of $\hat{A}(z,L)$.

strategy

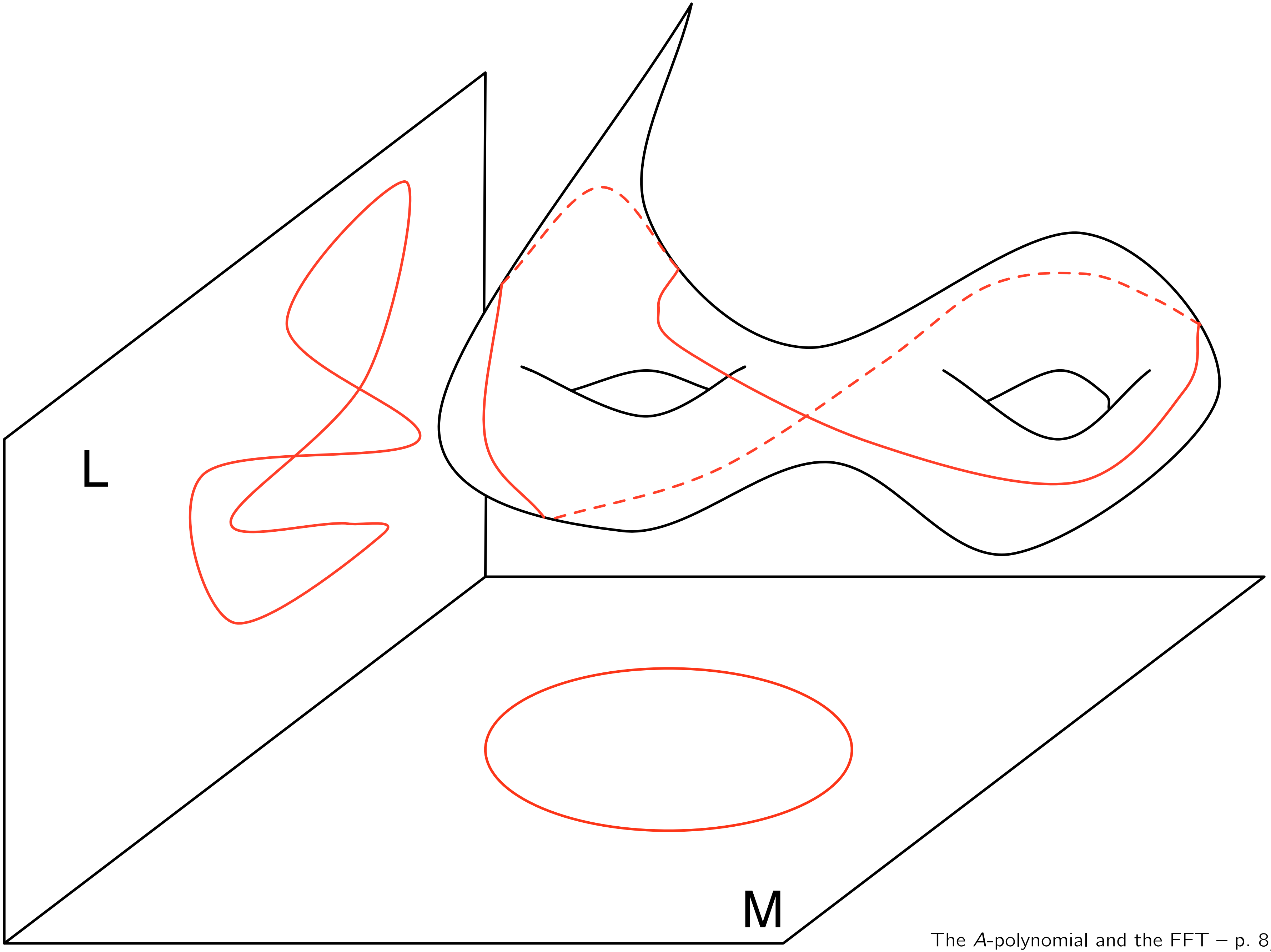
Write $A(M, L) = \sum_{n=0}^{D} a_n(M) L^n$.

Make it monic in L: $\hat{A}(M, L) = L^D + \sum_{n=0}^{D-1} \hat{a}_n(M) L^n$ where $\hat{a}_n(M) = a_n(M)/a_D(M)$. NOTE: often $a_D = M^k$, e.g. whenever ∞ is not a detected boundary slope. In these cases the $\hat{a}_n$ are Laurent polynomials.

Treat M as a parameter, so $\hat{A}(z, L) = \prod_{i=1}^{D} (L - \lambda_i(z))$, where $\{\lambda_1(z), \ldots, \lambda_D(z)\}$ are the roots of $\hat{A}(z, L)$.

First sample $\hat{a}_n(z)$ by computing elementary symmetric functions of $\{\lambda_1(z), \ldots, \lambda_D(z)\}$. Then use the FFT to find the coefficients of $\hat{a}_n(z)$.

strategy



computing $\lambda_i(z)$

computing $\lambda_i(z)$

Most of the work is done by Newton's method, so we must avoid singularities. It is guaranteed that there will be singularities above the unit circle in the M-plane, so we take z to run around a circle of randomly chosen radius $r \approx 1$. We renormalize at the end when we compute the coefficients.

computing $\lambda_i(z)$

Most of the work is done by Newton's method, so we must avoid singularities. It is guaranteed that there will be singularities above the unit circle in the M-plane, so we take z to run around a circle of randomly chosen radius $r \approx 1$. We renormalize at the end when we compute the coefficients.

Since we avoid singularities, the map $M : X(\mathcal{M}) \to \mathbb{C}$ looks like a covering map. The curves $\lambda_i(re^i\theta)$ are projections of lifts of the circle $|z| = r$. There are two steps in computing them.

computing $\lambda_i(z)$

Most of the work is done by Newton's method, so we must avoid singularities. It is guaranteed that there will be singularities above the unit circle in the M-plane, so we take z to run around a circle of randomly chosen radius $r \approx 1$. We renormalize at the end when we compute the coefficients.

Since we avoid singularities, the map $M : X(\mathcal{M}) \to \mathbb{C}$ looks like a covering map. The curves $\lambda_i(re^i\theta)$ are projections of lifts of the circle $|z| = r$. There are two steps in computing them.

- 1. Find the fiber over the point $z = r$. This means finding *all* solutions of a certain system of polynomial equations. We use Jan Verschelde's program PHC for this step.

computing λ_i(z)

Most of the work is done by Newton's method, so we must avoid singularities. It is guaranteed that there will be singularities above the unit circle in the M-plane, so we take z to run around a circle of randomly chosen radius $r \approx 1$. We renormalize at the end when we compute the coefficients.

Since we avoid singularities, the map $M : X(\mathcal{M}) \to \mathbb{C}$ looks like a covering map. The curves $\lambda_i(re^i\theta)$ are projections of lifts of the circle $|z| = r$. There are two steps in computing them.

- 1. Find the fiber over the point $z = r$. This means finding *all* solutions of a certain system of polynomial equations. We use Jan Verschelde's program PHC for this step.

- 2. Do covering space path-lifting to lift the circle $|z| = r$ to each point of the starting fiber. In the computational world path-lifting means Newton's method. We use SnapPea for this step.

time for a demo

property P

property P

Kronheimer's and Mrowka's proof of the property P conjecture implies:

if K is a non-trivial knot in S^3 then For every rational number $\frac{p}{q}$ with $|\frac{p}{q}| \leq 2$ there is a point $(L, M) \in \mathbb{C}^* \times \mathbb{C}^*$ such that $A_K(L, M) = 0$ and $M^p L^q = 1$

(These points correspond to irreducible $SU(2)$ representations.)

property P

Kronheimer's and Mrowka's proof of the property P conjecture implies:

if K is a non-trivial knot in S^3 then For every rational number $\frac{p}{q}$ with $|\frac{p}{q}| \leq 2$ there is a point $(L, M) \in \mathbb{C}^* \times \mathbb{C}^*$ such that $A_K(L, M) = 0$ and $M^p L^q = 1$

(These points correspond to irreducible $SU(2)$ representations.)

The proof depends on Gabai's "strong" Property R result, work of Thurston and Eliashberg linking contact structures and taut foliations, Eliashberg's embedding of 3-manifolds into symplectic 4-manifolds, Seiberg-Whitten theory (Taubes), results of Feehan and Leness relating Donaldson invariants to Seiberg-Whitten invariants, ...

property P

Kronheimer's and Mrowka's proof of the property P conjecture implies:

if K is a non-trivial knot in S^3 then For every rational number $\frac{p}{q}$ with $|\frac{p}{q}| \leq 2$ there is a point $(L, M) \in \mathbb{C}^* \times \mathbb{C}^*$ such that $A_K(L, M) = 0$ and $M^p L^q = 1$

(These points correspond to irreducible $SU(2)$ representations.)

Dunfield and Garoufalidis observe that this implies:

If K is a non-trivial knot in S^3 then the algebraic curve defined by A_K has 1-dimensional intersection with the unit torus in $\mathbb{C}^* \times \mathbb{C}^*$.

That is, some of the lifts $\lambda_i(\theta)$ meet the unit circle in arcs.

property P

Kronheimer's and Mrowka's proof of the property P conjecture implies:

if K is a non-trivial knot in S^3 then For every rational number $\frac{p}{q}$ with $|\frac{p}{q}| \leq 2$ there is a point $(L, M) \in \mathbb{C}^* \times \mathbb{C}^*$ such that $A_K(L, M) = 0$ and $M^p L^q = 1$

(These points correspond to irreducible $SU(2)$ representations.)

Dunfield and Garoufalidis observe that this implies:

If K is a non-trivial knot in S^3 then the algebraic curve defined by A_K has 1-dimensional intersection with the unit torus in $\mathbb{C}^* \times \mathbb{C}^*$.

That is, some of the lifts $\lambda_i(\theta)$ meet the unit circle in arcs.

Q: Is there a direct proof of this, based on the theory of $SL_2(\mathbb{C})$ character varieties of 3-manifolds.

Mahler measure

Mahler measure

The *Mahler measure* of a monic polynomial $f(x) \in \mathbb{Z}[x]$ is

$$m(f) = \prod_{f(\alpha)=0} \max(1, |\alpha|) = \exp \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

Mahler measure

The *Mahler measure* of a monic polynomial $f(x) \in \mathbb{Z}[x]$ is

$$m(f) = \prod_{f(\alpha)=0} \max(1, |\alpha|) = \exp \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

Lehmer's Question (1933) *Does there exist* $\delta > 0$ *such that* $m(f) > 1 + \delta$ *for every monic, irreducible and non-cyclotomic* $f(x) \in \mathbb{Z}[x]$*?* It is conjectured that $m(f) \geq 1.17628\ldots$ and that the extremal example is *Lehmer's polynomial*

$$L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

Mahler measure

The *Mahler measure* of a monic polynomial $f(x) \in \mathbb{Z}[x]$ is

$$m(f) = \prod_{f(\alpha)=0} \max(1, |\alpha|) = \exp \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

It turns out that $L(-x)$ is the Alexander polynomial of the $(-2, 3, 7)$ pretzel knot, which is extremal with respect to the properties of its character variety. (E.g. it has the maximal number (3) of cyclic Dehn surgeries.)

In general, every irreducible factor of the polynomial $A_K(M, 1)$ divides the Alexander polynomial of K.

Lehmer's polynomial is a *Salem polynomial*, i.e. it has exactly one root outside the unit circle. All Salem polynomials are palindromic, as are Alexander polynomials. It is known that any counterexample to Lehmer's question would have to be palindromic.

Mahler measure

The *Mahler measure* of a monic polynomial $f(x) \in \mathbb{Z}[x]$ is

$$m(f) = \prod_{f(\alpha)=0} \max(1, |\alpha|) = \exp \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

To define the Mahler measure of $f(x, y) \in \mathbb{Z}[x]$ replace the integral over the circle with the corresponding integral over the unit torus.

Mahler measure

The *Mahler measure* of a monic polynomial $f(x) \in \mathbb{Z}[x]$ is

$$m(f) = \prod_{f(\alpha)=0} \max(1, |\alpha|) = \exp \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

To define the Mahler measure of $f(x, y) \in \mathbb{Z}[x]$ replace the integral over the circle with the corresponding integral over the unit torus.

There is a well defined "volume function" V on the curve defined by $A(L, M)$, and $dV = \log |L| d \arg M - \log |M| d \arg L$.

Boyd and Rodriguez Villegas observe that $m(A(L, M))$ is the integral of dV over the subpaths of the lifts $\lambda_i(\theta)$ which lie outside of the unit circle. Thus $m(A(L, M))$ is a sum of volumes of certain $SL_2(\mathbb{C})$ representations of $\pi_1(\mathcal{M})$.

questions

questions

1. Does the (palindromic) polynomial $A(L, -1)$ factor in $\mathbb{Z}[x]$ as a product of cyclotomic polynomials, Salem polynomials, and irreducible polynomials with exactly one pair of conjugate roots outside the unit circle?

questions

1. Does the (palindromic) polynomial $A(L, -1)$ factor in $\mathbb{Z}[x]$ as a product of cyclotomic polynomials, Salem polynomials, and irreducible polynomials with exactly one pair of conjugate roots outside the unit circle?

2. (Agol) Is $m(A(L, M))$ equal to a sum of volumes of representations corresponding to characters $\chi \in X(\mathcal{M})$ such that $\chi(\mu) = 2$?