RESEARCH BLOG 9/21/04
BLACK OAK ARKANSAS CONJECTURE

Richard Kent pointed out that there is a much simpler way to see that the mapping class group of the n-punctured sphere does not have property T than what I was mentioning in blog 9/20/04. Namely, there is a finite index subgroup mapping onto $\mathbb{Z}$ (actually, one can get a map onto $\mathbb{Z} * \mathbb{Z}$, as Kent points out). Take the pure mapping class group (which doesn't permute punctures), which is of finite index in the mapping class group, and erase all but 4 points. This gives a map onto the 4 punctured mapping class group, which is virtually free.

Today, Pete Storm posted a couple of papers at the ArXiv, one of which gives a solution to a conjecture of Bonahon. This conjecture has also been formulated by Canary, Taylor and Minsky as the Black Oak Arkansas conjecture [3]. Apparently, Black Oak Arkansas is a band that Dick Canary was listening to when he thought of these conjectures (see http://www.blackoakarkansas.bigstep.com/). The conjectures state (roughly) that if one has a family of homeomorphic geometrically finite hyperbolic 3-manifolds N, then the volume of the convex cores $C(N)$ are minimized by $\frac{1}{2}$ the Gromov norm of the double along the boundary (times $V_3 = 1.01494$: Pete calls this the simplicial volume). If N is acylindrical, then $C(N)$ is minimal when $C(N)$ has totally geodesic boundary (there is a relative version as well, keeping track of rank 1 cusps). Pete had already proven this case in a previous paper. Bonahon had previously shown that the geodesic boundary case is locally minimal, making use of a generalized Schlafli formula for laminations [2], but he had trouble achieving a global result since the parametrization of convex hulls by bending measure of the boundary is difficult (it seems to be known only in the case of a punctured torus, by Keen and Series [5]). Pete's method is clever - he doubles $C(N)$ along its boundary, to obtain a manifold with a metric of Alexandrov curvature ≥ -1. This means that if one compares a triangle in the universal cover of the space with a triangle in $\mathbb{H}^2$, the points along the edge of a triangle will have distance to the opposite vertex $\geq$ the corresponding distance in $\mathbb{H}^2$. He generalizes the method of Besson, Courtois, and Gallot (BCG) [1] to the case of Alexandrov spaces. BCG have developed a very general version of Mostow rigidity, which has been very influential. Pete's main theorem is (roughly) that if one has a hyperbolic manifold of finite volume and constant curvature -1, then any metric of Alexandrov curvature ≥ -1 will have larger volume than the hyperbolic metric (by results of Otsu and Shioya [6], volume of Alexandrov metrics on manifolds is well-defined!). Actually, one need only know that the manifold has topological entropy $\geq$ the entropy of the hyperbolic metric. The topological entropy of a metric space with Hausdorff measure m is $\lim_{R\to\infty} \log(m(B_R(x)))/R$. By a result of Manning, this is well-defined and independent of x. For Alexandrov spaces on an n-manifold, Pete tells me that the Hausdorff measure is n-dimensional. The entropy of the manifold is the entropy of the universal cover. One might wonder why Pete has to go to all this effort to generalize BCG to the topological category? For a hyperbolic manifold N, one could choose a slightly larger convex core $C_\epsilon(N)$ which has smooth boundary, then double along it, smoothing out the metric. The difficulty is that it is hard to show that when $Vol(C(N_0)) = \inf Vol(C(N))$, then $C(N_0)$ has totally geodesic boundary. But in the non-acylindrical case, Pete uses this smoothing trick, then applies the thesis of Juan Souto [7] to show that the volume is almost $\geq$ the volume of the guts, and uses the fact that the "natural map" is uniformly Lipschitz. The "natural map" is obtained by using the visual measure in one metric to produce a metric on the sphere at ∞, then pushing forward to the hyperbolic space, then taking the barycenter. If one has a measure on $\partial_\infty \mathbb{H}^n$, which doesn't have atomic measure with support two points, then there is a well-defined barycenter. This is essentially because from any point in hyperbolic space, one can take the visual average of the measure on $\partial_\infty \mathbb{H}^n$, which gives a proper harmonic function on $\mathbb{H}^n$, so it has a unique minimum, which gives the barycenter. Souto computes the minimal volume for irreducible 3-manifolds satisfying the geometrization conjecture. If the manifold is not hyperbolic or Seifert fibered, then the diameter of the manifolds realizing the minimal volume with curvature ≥ -1 will have diameter $\to \infty$ as it stretches along tori, contradicting the Lipschitz condition for the natural map. So Pete gets a contradiction in this case, and reduces to the acylindrical case, where now he can use his Alexandrov version of BCG.

There are some conjectures which would generalize Pete's theorem which I would be interested in knowing. If we have a geometrically finite hyperbolic manifold N with acylindrical boundary, then we can find a compact core with embedded minimal surface boundary (which will lie inside the convex core). Thurston and I conjecture that the volume of these cores are minimized in the geodesic boundary case. One approach may be to use the monotonicity formula of Perelman, mentioned in blog 7/6/04. If we take the core with minimal surface boundary, and double it, smoothing off the metric so that the scalar curvature is arbitarily close to -6, and so that the volume is arbitrarily close to the volume of the doubled core, then the monotonicity estimate would imply this conjecture. One can make similar conjectures about volumes of harmonic cores, or generalized Gromov norms of geometrically finite manifolds. Basically, I want a notion of volume of geometrically finite manifolds which has three properties:

(1) It is minimized for the manifolds with geodesic boundary
(2) it is additive (or superadditive) under Maskit combination
(3) it gives the volume of the manifold in the finite volume case.

Then one could make some interesting conclusions about the behavior of the volumes of hyperbolic Haken 3-manifolds, improving volume estimates obtained in my thesis.

At Oberwolfach, Hyam Rubinstein reminded me about his approach to construct minimal surfaces in hyperbolic manifolds by barrier type arguments. He gave a series of talks at MSRI on minimal surfaces in 3-manifolds, and this construction is mentioned in his third talk. I pointed out to him that his method should give examples of hyperbolic 3-manifolds fibering over S^1 which have no foliation by minimal copies of the fiber. By a result of Sullivan, given any taut foliation, one may find a metric in which all the leaves are minimal. But the question is, can this metric be hyperbolic? This question was posed by

Uhlenbeck [8]. Rubinstein's construction in this case proceeds by taking a hyperbolic 3-manifold fibering over S^1 and drilling out a simple closed curve on the fiber, which has a complete hyperbolic metric of finite volume on the complement. Performing a high enough Dehn twist about this curve corresponds to doing a high Dehn filling on the curve complement (with correct coefficient). Since there is a (noncompact embedded) minimal surface in the drilled manifold which is the fiber surface with the curve removed, one should be able to insert a minimal annulus with boundary on this surface, and the boundary curves parallel to the drilled curve. I think one may choose this annulus to come from the unique complete minimal minimal surface which is invariant by a 1-parameter family of parabolic isometries (see [4]). One uses this to slice off the ends of the minimal surface going into the cusp. The resulting surface is piecewise minimal and isotopic to the fiber, but not minimal. Also, the angle between each minimal piece of the surface is $< \pi$ on the side in which the annulus was inserted. Then a high Dehn filling changes the geometry very little, so this piecewise minimal surface should persist. But a minimal foliation would have to be tangent to this surface at some point, in which case it would coincide, by the maximum principle, contradicting the fact that the surface is piecewise minimal. What I don't really understand here is why one can find nearby piecewise minimal surfaces under Dehn filling.

References

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