RESEARCH BLOG 8/31/04

John Berge wrote me mentioning that there was a mistake in blog 6/8/04, in the definition of a primitive curve in the boundary of a handlebody. A primitive curve is supposed to represent a conjugacy class of a generator in the free fundamental group of the handlebody. Thus, the correct definition is that the curve is primitive if there is a meridian disk for the handlebody which intersects the curve exactly once. I mistakenly said that, for a curve on the boundary of a genus 2 handlebody, a primitive curve is defined as being disjoint from a meridian disk (which is only a necessary condition). In fact, if this holds, then the curve will be a multiple of a generator in the fundamental group of the handlebody (this is special to the genus 2 case). Maybe in this case, this condition should be called *multiprimitive*. Adding a handle along a multiprimitive curve results in a connect sum of a lens space and a solid torus (where we allow S^3 and $S^2 \times S^1$ to be lens spaces). If a knot in a genus 2 Heegaard surface of a manifold is multiprimitive in each handlebody, then surgery along the framing induced from the Heegaard surface gives a manifold which is a connect sum of 3 lens spaces (again, some of the lens spaces could be trivial). If the knot lies in a genus 2 Heegaard splitting of S^3, then the cabling conjecture would imply that if the knot is doubly multiprimitive, then it is a torus knot, or else the knot must be primitive in one handlebody or the other. But I don't claim to have investigated this too carefully: there may be simple reasons a hyperbolic knot in S^3 cannot be doubly multiprimitive, e.g. homology. Anyway, this might be an interesting special case of the cabling conjecture to investigate. When doing a web search on the cabling conjecture, I found notes from a talk by Jim Howie. He shows that the group $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z} * \mathbb{Z}/r\mathbb{Z}$ has weight ≥ 2, which means that if one adds one relator to this group, it is still non-trivial (see his preprint for a more precise statement), answering a conjecture of Scott-Wiegold.

This implies that if one does non-trivial surgery on a knot in S^3 to obtain a reducible manifold, then there are at most 3 prime factors, and two of these are lens spaces, the third is a homology sphere.

I mentioned in blog 7/21/04 that Kellerhals had published a note proving that the minimal volume 4-orbifold is a certain quotient of $\mathbb{H}^4$ by a Coxeter group. Since then, she has retracted her paper, and it will not be published. Dick Canary wrote me, asking about a part of her argument that he did not understand, and I was not able to understand it as well. I suspect that one may be able to fix her argument by considering collaring estimates for finite non-abelian subgroups of a lattice in $Isom(\mathbb{H}^4)$, analogous to the work of Gehring and Martin in the 3-dimensional case. The idea here is that if one places a maximal precisely invariant sphere at a fixed point of this group (precisely invariant means that it is fixed by the finite subgroup, and does not intersect its image under any element of the lattice outside of this finite subgroup; in the quotient space, the quotient of the ball by the finite group is an embedded suborbifold of the 4-orbifold), then one may estimate the number of other spheres kissing it, by considering the orders of cosets of maximal cyclic subgroups of the finite group which fix an axis. Then one should be able to apply kissing number estimates to get a lower bound on the radius of this sphere. The reason that one might expect this approach to work, is that the Coxeter group Kellerhals considers realizes Boroczky's sphere packing bounds, and thus is quite special.

I've been trying to learn something about arithmetic 3-manifolds. As I mentioned in blog 8/18/04 , I would like to show that there are only finitely many closed arithmetic hyperbolic 3-manifolds with 2-generator fundamental group. When I explained to Alan Reid my result that there are only finitely many 2-generator closed hyperbolic manifolds which have injectivity radius $> \epsilon$, and are not Heegaard genus 2, Alan wondered whether one might be able to apply this to study two-generator arithmetic 3-manifolds. He pointed out that the Salem conjecture would imply that a closed arithmetic 3-manifold has a lower bound on the injectivity radius. The idea here is that the fundamental group of an arithmetic 3-manifold can be considered as a subgroup $\Gamma < \mathrm{PSL}_2(\overline{\mathbb{Q}}) \leq \mathrm{PSL}_2\mathbb{C}$ (such that Γ has integral traces).

If one takes an element of $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, which does not fix $\{\Gamma, \overline{\Gamma}\}$ (acting naturally on matrix coefficients), then $\sigma(\Gamma) < SU(2)$. Thus, the non-trivial Galois conjugates of any matrix in Γ will have eigenvalues which are roots of unity (complex conjugation is considered a trivial Galois autmorphism, even though it may not preserve Γ). So if λ is an eigenvalue of a matrix of Γ, then its Galois conjugates will be $\subset \{\lambda^{-1}, \overline{\lambda}, \overline{\lambda^{-1}}\} \cup S^1$. Thus, $|\lambda|^2$ will be a Salem number: it is a real number α, whose Galois conjugates are $\subset \{\alpha^{-1}\} \cup S^1$. The Salem number conjecture implies that there is a minimal Salem number, and in fact it is conjectured to be the maximal real root of Lehmer's polynomial

$$L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

This is a special case of Lehmer's conjecture, which implies that this number minimizes the Mahler measure over all algebraic numbers. In any case, it is clear that the Salem number conjecture implies that there is a lower bound on the length of a curve in an arithmetic 3-manifold.

Thus, assuming the Salem number conjecture, all but finitely many closed arithmetic hyperbolic 3-manifolds with 2-generator fundamental group will have Heegaard genus 2. Now, given an arithmetic lattice $\Gamma < \mathrm{PSL}_2\mathbb{C}$, there is a maximal arithmetic group $\Gamma < \Gamma' < \mathrm{PSL}_2\mathbb{C}$. The maximal arithmetic groups were described by Borel [2]. For the following discussion, I'm going to assume that one understands a bit about the way that arithmetic groups are defined, see section 11.4 of the book by MacLachlan and Reid. Given a quaternion algebra A over a field k, a maximal arithmetic group comes from an order $\mathcal{O} \subset A$ which is either a maximal order, or an intersection of two maximal orders. One takes the normalizer in A of $\mathcal{O}$, and then takes the subgroup of invertible elements of this normalizer, then projectivizes to get a subgroup of $\mathrm{PSL}_2\mathbb{C}$. So any arithmetic lattice $\Gamma \subset \mathrm{PSL}_2\mathbb{C}$ will be a subgroup of such a maximal discrete subgroup of invertible elements in A. The key fact that we need is that this group is a congruence subgroup associated to A (this was explained to me by Alan Reid). If one takes $\mathcal{O}^{(1)}$, the elements in $\mathcal{O}$ of norm 1, and a two-sided ideal $I \subset \mathcal{O}$, then a *principal congruence subgroup* consists of elements of $\mathcal{O}^{(1)} \cap (1+I)$. A *congruence subgroup* is a subgroup of A which contains a principal congruence subgroup of A, for *some* maximal order $\mathcal{O} \subset A$. Since for two maximal orders $\mathcal{O}_1, \mathcal{O}_2$, $\mathcal{O}_1 \cap \mathcal{O}_2$ is a two-sided ideal in $\mathcal{O}_i$, then one observes that all maximal arithmetic groups are congruence groups (we are here using the fact that $1 + \mathcal{O} = \mathcal{O}$, since $1 \in \mathcal{O}$).

Now, we turn to a deep number theoretic result, which is related to the generalized Ramanujan conjectures, which I learned about from a talk of Alan Reid in Arkansas (see blog 4/22/03). The result states that for arithmetic congruence subgroups $\Gamma < \mathrm{PSL}_2\mathbb{C}$, there is a universal lower bound on $\lambda_1(\mathbb{H}^3/\Gamma)$, where λ_1 is the minimal non-zero eigenvalue of the Laplacian Δ (which is > 0 since the Laplacian has non-negative eigenvalues). In fact, one need only prove this for principal congruence subgroups, since if $\Gamma < \Gamma'$, then clearly $\lambda_1(\mathbb{H}^3/\Gamma) > \lambda_1(\mathbb{H}^3/\Gamma')$, since by the covering map $\mathbb{H}^3/\Gamma \to \mathbb{H}^3/\Gamma'$, eigenfunctions on $\mathbb{H}^3/\Gamma'$ lead to eigenfunctions on $\mathbb{H}^3/\Gamma$. Selberg proved that for congruence subgroups $\Gamma < \mathrm{PSL}_2\mathbb{Z}$, $\lambda_1(\mathbb{H}^2/\Gamma) \geq \frac{3}{16}$, and he conjectured that $\lambda_1(\mathbb{H}^2/\Gamma) \geq \frac{1}{4}$. This is known as the Selberg conjecture. Selberg's conjecture has a natural generalization to any arithmetic group, and this conjecture has been subsumed by the generalized Ramanujan conjecture, a conjecture about representations of algebraic groups (Ramanujan's original conjecture was solved by Deligne, and is about the coefficients of automorphic forms, which seems at first to have no relation to spectra of manifolds! See Sarnak's Fields Institute lectures). Vignéras wrote an article [6] demonstrating how Selberg's theorem follows from results of Jacquet-Langlands [5] and Gelbart-Jacquet [4] (which proves a special case of the generalized Ramujan conjectures). She also explained how the same argument holds for any congruence arithmetic fuchsian surface whose trace field is $\mathbb{Q}$. But apparently it is a folklore theorem that in fact this argument works for any arithmetic congruence subgroup of $\mathrm{PSL}_2\mathbb{C}$ (Sarnak and Vignéras both responded to e-mails from me confirming this). I still haven't understood how this all works though.

Now, how is the lower bound on $\lambda_1(\mathbb{H}^3/\Gamma)$ relevant to our question? We relate this to geometric information by using an inequality of Buser [3]. The better known Cheeger inequality states that $\lambda_1(M) \geq$

$h^2(M)/4$, where $h(M)$ is the isoperimetric constant (or Cheeger constant) for a Riemannian manifolds M, given by

$$h(M) = \inf\{\max\{\mathrm{Vol}(S)/\mathrm{Vol}(A), \mathrm{Vol}(S)/\mathrm{Vol}(B)\} | M = A \cup_S B\},$$

where A and B are submanifolds such that $A \cap B = S = \partial A = \partial B$. Buser's inequality is a sort of converse (we state only a special case). If M^n is a hyperbolic manifold, then $\lambda_1(M) \leq 2(n-1)h(M) + 10h^2(M)$. Thus, $\lambda_1(M) \to 0$ if and only if $h(M) \to 0$, for a family of hyperbolic manifolds of fixed dimension. Buser's inequality also naturally extends to hyperbolic orbifolds. Thus, minimal arithmetic 3-manifolds have a universal lower bound on the Cheeger constant, given the above mentioned lower bound on λ_1.

Finally, we may apply this result to arithmetic 3-manifolds with 2-generator fundamental group. Suppose we have an infinite sequence of closed arithmetic 3-manifolds M_i such that $\pi_1(M_i)$ is 2-generator. Assuming the Salem number conjecture, all but finitely many of these will have Heegaard genus 2. Now, for a manifold of Heegaard genus g, there is a sweepout by simplicial ruled surfaces of genus g [1], such that the degree of this sweepout is one. Each M_i covers a minimal arithmetic orbifold M_i'. Since an indecomposable 2-generator hyperbolic 3-manifold has a bounded generating set, a fixed orbifold group has only finitely many 2-generator subgroups of finite index. Thus, we may assume that the M_i's are distinct, and therefore the volumes of $M_i' \to \infty$, since volumes of arithmetic orbifolds are discrete. When we project the sweepout by simplicial ruled surfaces of genus 2 from M_i to M_i', each surface will project to a surface of bounded area in M_i'. At one end of the sweepout, the surface bounds a very small volume handlebody on one side in M_i (which approximates a graph), and at the other end of the sweepout bounds a very small handlebody on the other side. Thus, the same property holds for the sweepout by surfaces projected to M_i'. Thus, at some point during the projected sweepout in M_i', the surface will divide M_i' in half (one can do oriented cut and paste to get an embedded surface). Since $\mathrm{Vol}(M_i') \to \infty$, we see that $h(M_i') \to 0$, a contradiction.

References

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