RESEARCH BLOG 8/18/04
CONJECTURES OF WALDHAUSEN AND SEDGWICK

Tao Li posted two papers on the ArXiv which claim to solve some conjectures about Heegaard splittings. The first paper solves the Waldhausen conjecture. This conjecture states that for any $g \in \mathbb{N}$, an atoroidal 3-manifold M has only finitely many Heegaard splittings of genus $\leq g$. A solution was claimed a while ago by Jaco and Rubinstein, using their tool of 1-efficient triangulations, but has yet to appear. They show that given a closed irreducible 3-manifold, one may find a triangulation which has only one normal 2-sphere (linking the vertex), and only finitely many normal tori. Then the proof of the Waldhausen conjecture follows by a standard argument using almost normal branched surfaces (which I alluded to last blog). First, one may assume that the Heegaard splittings are strongly irreducible by detelescoping. If there is an infinite number of strongly irreducible Heegaard splittings of genus $\leq g$, then one may take a subsequence of them which are almost normal and fully carried by a fixed almost normal branched surface. Taking differences between the normal coordinates, one obtains a normal surface of Euler characteristic zero, which therefore must consist of copies of the finitely many normal tori. Then by isotopy, they show that one may find a smaller weight almost normal representative of the Heegaard splittings, which gives a contradiction. Tao Li gets around using efficient triangulations by showing that one may split a branched surface so that it carries no tori, but it still carries the Heegaard surfaces carried by the original branched surface. I'm not sure how his argument goes, but it sounds plausible. In our paper giving an algorithm to recognize laminar 3-manifolds, we made use of 1-efficient triangulations, but Tao's new method should apply in that situation as well.

His second paper claims to solve a conjecture of Sedgwick, which I mentioned last time. One way to say this is that a small 3-manifold has only finitely many strongly irreducible Heegaard splittings. Again, he proceeds by using branched surfaces, but I haven't read enough yet to know what the main ideas are.

I've worked out some ideas on 2-generator hyperbolic manifolds. I believe that I can show that given $\epsilon > 0$, there are only finitely many hyperbolic 3-manifolds which have 2-generator fundamental group, injectivity radius bounded below by ϵ, and Heegaard genus > 2. I conjecture that all 2-generator hyperbolic 3-manifolds have Heegaard genus 2. The main tool is the orbifold drilling technique of Canary and the tameness conjecture. First, one shows that for $\epsilon > 0$, if M is a hyperbolic 3-manifold with 2-generator fundamental group, and injectivity radius $> \epsilon$, then there is a rank two graph in M which carries $\pi_1 M$, and has bounded length, where the bound only depends on ϵ. This generalizes an argument of Gromov and Delzant for two-generator indecomposable word-hyperbolic groups. Then one takes a geometric limit of an infinite sequence of these groups, where the basepoint is chosen somewhere near the graph of bounded length. The limit is a 2-generator simply degenerate free Kleinian group. There are genus two simplicial ruled surfaces which exit the end of this limit manifold. We can push these back to the approximates, and then show that they are Heegaard surfaces. Take a diskbusting geodesic, and push it back to the approximates, and take the genus 2 surfaces which are very far away. Then do π orbifold drilling along the geodesic. If the surface is compressible, then in fact it must be a Heegaard surface. If it is incompressible, the we pass to a cover of the orbifold corresponding to the fundamental group of the suborbifold bounded by the surface and containing the orbifold locus. The boundary of the convex core of this cover will project to a bounded neighborhood of the orbifold locus. When we interpolate pleated surfaces, then they will sweep through the whole manifold. One then applies the compactness theorem for pleated surfaces to get a contradiction.

I believe I can apply this result to show that there are only finitely many 2-generator arithmetic hyperbolic manifolds, at least assuming the Salem number conjecture and the generalized Ramanujan conjecture. I'll try to describe these next blog.