Recently, Moriah, Schleimer, and Sedgwick posted a paper which is aimed at resolving a conjecture of Sedgwick's about Heegaard splittings. Namely, Sedgwick conjectured that an irreducible 3-manifold with infinitely many distinct (non-isotopic) strongly irreducible Heegaard splittings is Haken. The motivation for this conjecture is that the known examples of manifolds with infinitely many strongly irreducible Heegaard splittings (first discovered by Casson and Gordon) are Haken. The splittings are obtained from a minimal genus splitting by "adding" copies of a particular incompressible surface. If one takes two (orientable, coorientable) surfaces in a 3-manifold which are transverse, there is no canonical way to add them, by cutting and pasting, to get another embedded surface, since for each intersection curve, there are two ways to resolve the intersection. But once one has made one such choice for each intersection curve, one may add arbitrarily many copies of the pair of surfaces together so that the choice of resolution is always the same. Another motivation for Sedgwick's conjecture, is that if one had an infinite collection of strongly irreducible Heegaard splittings, one could put them in almost normal form with respect to a fixed triangulation, by a result of Rubinstein and Stocking. Taking an infinite sequence of these, one may choose a subsequence such that the differences between almost normal surfaces give normal surfaces, which are fully carried by a fixed branched surface. To prove the Sedgwick conjecture, one would like to prove that this branched surface is incompressible, in which case the surfaces carried by it would be incompressible. However, this seems to be difficult to carry out, in particular how to use the strong irreducibility to prove that the branched surface is incompressible. What they have done is a special case, to assume that the branched surface is actually a surface K, being added to a strongly irreducible Heegaard splitting H to get strongly irreducible surfaces $H+nK$, where "+" indicates cut and paste addition for some fixed choice of resolutions (one may imagine that these come from normal addition with respect to a triangulation). A key tool that they use is Scharlemann's no nesting lemma: if one has a strongly irreducible Heegaard surface, then the only unknots which lie on the Heegaard surface and have framing zero are meridian disks (this is the no-nesting property, because an unknot which is not a meridian disk would bound a disk whose interior would intersect the Heegaard surface in nested circles). This gives a close analogy with incompressible surfaces. I believe that the converse is true as well: any surface with the no nesting property is strongly irreducible (or incompressible). Their argument proceeds by assuming that the surface K being added to a strongly irreducible splitting H is compressible, and then using the compressing disk for K to find a disk bounding $H + nK$ (for large enough n) which violates the no nesting lemma.

The no nesting lemma seems somewhat reminiscent of the notion of a tight contact structure. A contact structure on a 3-manifold is a plane field which is nowhere integrable. A knot which is tangent to the plane field (a Legendrian knot) gets an induced framing. If one has a Legendrian unknot with framing zero, then the contact structure is overtwisted; if no such unknot exists, then the contact structure is tight. Maybe there is a theory (such as confoliations) which connects the notion of a strongly irreducible Heegaard splitting with the notion of a tight contact structure?

In fact, there is no known algorithm to detect if a Heegaard splitting is strongly irreducible (or irreducible, for that matter). There are certain conditions which imply that a Heegaard splitting is strongly irreducible, obtained by Casson. Waldhausen showed that an incompressible surface is strongly irreducible if it can be extended to a special hierarchy; the special hierarchy certifies that the surface is incompressible. Similarly, Gabai showed that a Thurston norm minimizing surface may be extended to a taut sutured manifold hierarchy which certifies that it is taut, and Li showed that a lamination is essential if and only if it is carried by an laminar branched surface. It would be interesting if one could find analogous notions for tight contact structures and for strongly irreducible Heegaard splittings. That is, one would like some sort of structure (such as a generalization of Casson's criterion) which would imply that a Heegaard splitting is strongly irreducible, and such that every Heegaard splitting is associated to such a structure. One may certify that certain contact structures are tight, by showing that they are symplectically fillable, but there are tight contact structures which are not symplectically fillable. A machinery of convex surfaces for proving tightness is being developed by Giroux, Honda, Kazez, and Matic, but I believe that they haven't obtained an algorithm for detecting tight contact structures yet, or shown that their conditions are necessary.