RESEARCH BLOG 7/19/04

A paper is published by Ruth Kellerhals, which proves that the minimal volume hyperbolic 4-orbifold is that $Q = H^4/\Gamma$, where Γ is the cocompact arithmetic Coxeter group $\circ\text{---}\circ\text{---}\circ\text{---}\circ\overset{5}{\text{---}}\circ$. The volume of Q equals $\pi^2/10,800$. This corroborates the phenomenological observation that minimal volume hyperbolic objects tend to be arithmetic. Since in even dimensions, the volume of a hyperbolic orbifold is proportional to its Euler characteristic, it is natural to wonder whether this orbifold has minimal Euler characteristic among aspherical "good" 4-orbifolds (ones with an aspherical manifold universal cover) with positive Euler characteristic (it is natural to conjecture that the Euler characteristic of such orbifolds is always non-negative). It is also natural to wonder whether the set of Euler characteristics of aspherical good $2n$-orbifolds is well-ordered. Certainly this is true in two dimensions, and seems to be consistent with what is known in higher dimensions, *e.g.* the examples of Gromov and Thurston. These examples are pinched negatively curved 4-orbifolds, which are obtained by introducing an order n cone singularity along a geodesic surface in a closed hyperbolic 4-manifold. As n approaches infinity, the Euler characteristic approaches (from below) that of the manifold with the geodesic surface removed.

Benson Farb wrote to me explaining how to solve the generalized word problem (GWP) in 3-manifolds with a geometric decomposition, answering a question I brought up in blog 6/14/04. In his paper on the generalized word problem, Farb introduces the concept of *distortion* of a subgroup. Let G be a finitely generated group, and $H < G$ be a finitely generated subgroup. We may take a set of generators $X \cup X^{-1}$ for G which includes generators $Y \cup Y^{-1}$ for H, and let $\|w\|_G$ denote the minimal length representative of $w \in G$ in the Cayley graph of G with respect to the generators $X \cup X^{-1}$. Similarly for $z \in H$, $\|z\|_H$ is the minimal expression of z as a word in $Y \cup Y^{-1}$. Then $f : \mathbb{N} \to \mathbb{N}$ is a *distortion function* for H in G if for every $z \in H$, $\|\overline{z}\|_H \leq f(\|\overline{z}\|_G)$.

Then one says that H has distortion f in G. Farb showed that the generalized word problem for H in G is solvable if and only if $H < G$ has a recursive distortion function f Now, I'll let Benson take over:

Anyway, when I was finishing as a grad student I came up with an argument to solve the generalized word problem for all freely indecomposable subgroups of compact 3-manifolds satisfying Thurston's conjecture.

I think the point is that it is enough to do it for every finite volume geometric 3-manifold, because: one needs to prove recursive distortion of a given subgroup H. *Now:*

1. If H *is a subgroup of a free product, we can write* H *as a free product of a free group and subgroups of conjugates of the factors. Knowing it for the factors then easily gives it for the whole thing (note that a factor in a free product is quasiconvexly embedded in that free product). So, we can assume the 3-manifold is irreducible.*

2. Now to deal with the torus decomposition. Well, note that whenever you are gluing 2 pieces in the torus decomposition, the tori along which you are gluing is quasi-convex in each factor (since you are either a cusp in a hyperbolic 3-manifold, or a cusp in an $\mathbb{H}^2 x \times \mathbb{R}$ *manifold (note if you have* $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ *structure but are noncompact then you have an* $\mathbb{H}^2 \times \mathbb{R}$ *structure). So, if we know it for each geometric piece, a slight generalization of the above argument in (1) gives it.*

Now to the geometric pieces. The hard case is the hyperbolic one. Thurston-Bonahon gives, for freely indecomposable H*, that you are geometrically finite (hence quasiconvex - done), or virtually a fiber (so distortion is exponential, in particular recursive - done).*

So, the real hard case is the freely decomposable case. I could never do this, and asked about it for years, but no luck. The above kind of argument is enough to prove the so-called "power problem" for every 3-manifold M *satisfying Thurston's conjecture: given elements* $u, v \in \pi_1 M$*, does some power of* u *equal* v*? There are groups with solvable word but unsolvable power problem (Collins).*

But, tameness does all hyperbolic 3-manifolds. So, I think that's it for the GWP for all compact 3-manifolds satisfying Thurston's conj. (which presumably will be proven soon to be everything).

I agree it would be neat to understand the complexity of this problem. Also note that the GWP is false for a product of 2 noncyclic free groups (Mihailova, 1950's); in fact Gersten (answering a conjecture of mine) found a finitely PRESENTED subgroup $H < F_2 \times F_2$ *which is has unsolvable GWP. So even for very simple 4-manifolds this problem is unsolvable!*

Best, Benson