RESEARCH BLOG 7/6/04
VOLUME ESTIMATES VIA RICCI FLOW

An article on Perelman's work on the geometrization conjecture was published in Scientific American. It gives an introduction to the subject, but doesn't have any new information or interviews.

In blog 8/20/03 I mentioned how one could improve upon the volume estimates I gave in [1] using Ricci flow. I gave a talk about this at a BIRS workshop last September. After the workshop, Nathan Dunfield figured out how to improve on the volume estimate. Now we can show that the minimal volume orientable hyperbolic 3-manifold has volume $> .649$ (dependent on Perelman's work). I also realized that there is a simpler monotonicity argument than the one I used (which was first discovered by Hamilton, and extends to Ricci flow with surgery, see *e.g.* Anderson's notes). Given M^3 closed and (M_t, g_t) a solution to the Ricci flow with surgery (where $M_0 = M$, and the topology of M_t is locally constant, but may change at the surgery times), the quantity $R_{min}V^{2/3}$ is non-decreasing with t when $R_{min} < 0$ (it follows from Perelman's claims that if M has a hyperbolic piece in its characteristic submanifold decomposition, then $R_{min} < 0$). For solutions to Ricci flow, this is proven via the maximum principle and the equations $\frac{d}{dt}R = \Delta R + 2|Ric|^2$ and $\frac{d}{dt}d\mathrm{Vol} = -Rd\mathrm{Vol}$ (see Ben Chow's notes on the maximum principle). During a surgery as defined in Perelman's second paper, the volume decreases, whereas R_{min} is unchanged (if negative), since the surgery occurs in a region of M where $R > 0$ before and after the surgery. If M is hyperbolic, then as $t \to \infty$, g_∞ approaches the hyperbolic metric on M, so we have

$$R_{min}V(g_0)^{2/3} \leq R_{min}V(g_\infty)^{2/3}.$$

Thus, if we take a hyperbolic 3-manifold (M, g_∞) (normalized so that the sectional curvature is -1), the scalar curvature is -6. If we choose a different metric (M, g_0), such that $R_{min}(g_0) \geq -6$, then from the above inequality, we have $V(g_0) \geq V(g_\infty)$. Thus, the hyperbolic metric on M infimizes volume over all metrics with scalar curvature pinched below by -6.

To see the improved volume estimate one obtains, one follows the argument of [1]. Assume that (M, g_∞) has a geodesic γ with an embedded tube of radius ρ at least $(\log 3)/2$ (by the main result of [2], this is true for all but finitely many hyperbolic 3-manifolds, and the exceptions all have volume > 1, which is bigger than .9427..., the volume of the Weeks manifold, which has minimal known volume). We may remove this tube, and replace it with a "horotube", which is isometric to a horosphere in $\mathbb{H}^3$ modulo $\mathbb{Z}^2$, which sectional curvature $-\kappa^2$. The resulting metric can be made C^0, but is not C^1. We need to choose κ so that the metric may be smoothed, while keeping R bounded from below. Then we compare the smoothed metric with the complete hyperbolic metric on $M-\gamma$ using the monotonicity formula. The smallest volume orientable cusped hyperbolic 3-manifold is the figure eight knot complement and its sibling which have volume 2.0298.... Thus, we may obtain a lower bound on the volume of the closed manifold via this comparison.

It turns out that the optimal choice for κ seems to be the mean curvature of the boundary of the tube of radius ρ about γ. Roughly, this is due to the fact that the sectional curvature is proportional to the leading term of the derivative of the principal curvature in the normal direction. Thus, when we interpolate between the two metrics, the scalar curvature (which is the sum of sectional curvatures) will remain bounded from below if we choose κ slightly larger than the mean curvature.

References

[1] I. Agol. Volume change under drilling. *Geom. Topol.*, 6:905–916 (electronic), 2002.

[2] D. Gabai, R. Meyerhoff, and N. Thurston. Homotopy hyperbolic 3-manifolds are hyperbolic. preprint, to appear in Annals of Mathematics,.