RESEARCH BLOG 6/14/04
TAMENESS IN TEXAS

Two weekends ago, Jeff Brock organized a conference Tameness in Texas, which was centered around the classification of Kleinian groups. I have a preprint available, which shows that every hyperbolic manifold with finitely generated fundamental group is homeomorphic to the interior of a compact manifold with boundary (so the ends are homeomorphic to a closed surface $\times\mathbb{R}$), answering the Marden conjecture (also called tameness), see blog 11/12/03 . As it happened, Calegari and Gabai have a proof as well, and Gabai gave talks at UT Austin the Thursday before the conference, as well as at the conference. In fact, Suhyoung Choi also has an argument, making use of one idea outlined in blog 11/14/03, but otherwise independent of my proof. In fact, all three arguments have very similar structure and ideas. Calegari and Gabai's proof is remarkable, in that they make no use of Bonahon or Canary's previous work on tameness in the indecomposable case. Their argument makes use of a notion of "shrinkwrap", which is a surface which is made minimal with geodesic barriers, and has an intrinsic $CAT(-1)$ structure. This replaces pleated surfaces introduced by Thurston or simplicial surfaces introduced by Bonahon and Canary. It has the distinct advantage that the surfaces may be approximated by embedded surfaces (in certain cases), and therefore may be used as barriers for further shrinkwrapping. I have a further application of my techniques using branched covers to the study of rank versus Heegaard genus for hyperbolic 3-manifolds, but I don't see how to use shrinkwrap to accomplish it yet.

Another application of tameness is to some group theory questions. The generalized word problem says, given a finitely presented group G, and a finitely generated subgroup $H < G$, along with an element $g \in G$, is $g \in H$? It appears that tameness implies that this problem is solvable for $G = \pi_1 M^3$, where M is a closed hyperbolic 3-manifold. The covering theorem of Canary/Thurston implies that if $H < G$ is finitely generated, then the corresponding cover $N \to M$ such that $H = \pi_1 N$ is either geometrically finite, or associated to a virtual fiber over S^1. In the second case, there is a finite index subgroup $M' \leq M$ which fibers over S^1, so that the fiber subgroup is contained in H with index ≤ 2. By a result of Farb [1], if $H < G$ is quasiconvex, then the generalized word problem for H is solvable. A subgroup $H < G$ is quasiconvex if for every element of H, the geodesic representative of any element of H in the Cayley graph of G lies in a bounded neighborhood of the subgraph corresponding to H (where we've thrown a set of generators of H into the generating set of G to form the Cayley graph). Quasi-convexity is preserved up to quasi-isometry for Gromov-hyperbolic spaces. Thus, if H is a geometrically finite subgroup of G, then H is quasi-convex in G, and therefore H has solvable generalized word problem in G. If H is geometrically infinite, then (assuming for simplicity it's a virtual fiber subgroup), one may first solve the generalized word problem for $\pi_1 M' \leq G$, since this group is of finite index. If $g \in \pi_1 M'$, then we may see if g lies in the kernel of the homomorphism $\pi_1 M' \twoheadrightarrow \mathbb{Z}$, to determine if $g \in H$. This answers a question posed in Farb's paper (at least in the closed case). It could be interesting to study the computational complexity of this question.

Another problem which seems to be solved by tameness is the presentation problem: given G, and a collection of elements $\{g_1, ..., g_n\} \subset G$, find a presentation for the group $H = \langle g_1, ..., g_n \rangle$ (this makes sense, since 3-manifold groups are coherent by Scott). If H is geometrically finite, then one should be able to search for a fundamental domain for $H \leq Isom(\mathbb{H}^3)$, using an algorithm of Riley (see the argument of Casson and Manning [2]). Otherwise H represents a surface subgroup associated to a virtual fiber. One may run another procedure which searches for an element $h \in G$ which normalizes H, and such that $H' = \langle h, H \rangle$ has finite index in G (which may be determined by ad hoc methods). The finite cover $M' \to M$ such that $\pi_1 M' = H'$ fibers over S^1 with fiber subgroup H. Then one finds an expression of the fiber subgroup in terms of the given generators of H to find a presentation.

The algorithms I've outlined are probably not very efficient. Also, I've overlooked how to search for an algorithm solving the generalized word problem given $H < G$ - I've only explained the existence of an algorithm. It would be interesting to analyze the computational complexity of these problems. Also, it would be nice to see if there are algorithms for these problems for general 3-manifolds satisfying the geometrization conjecture.

At the tameness conference, Canary discussed applications of tameness to various questions about Kleinian groups and some 3-manifold problems. Ken Bromberg gave a nice talk showing that the space of punctured torus groups with the topology of algebraic convergence is not locally connected. I won't describe his argument, but it comes down to the fact that if one takes a closed connected set $B \subset [0, \infty) \times \mathbb{R}$ which is invariant under $(x, y) \to (x, y + 1)$, so that ∂B is not a straight line, and a similar set $C \subset (-\infty, 0] \times \mathbb{R}$, then if one takes the minimal number r such that $B \cap C + (0, r) \neq \emptyset$, then $B \cap C + (0, r)$ is disconnected.

There were also talks by Bonahon, Myers, Gabai (which I mentioned already), and Souto. Hopefully I'll have some time to discuss these in later blogs.

References

[1] B. Farb. The extrinsic geometry of subgroups and the generalized word problem. *Proc. London Math. Soc. (3)*, 68(3):577–593, 1994.

[2] J. Manning. Algorithmic detection and description of hyperbolic structures on closed 3-manifolds with solvable word problem. *Geom. Topol.*, 6:1–25 (electronic), 2002.