RESEARCH BLOG 6/9/04
EULER CHARACTERISTIC OF MANIFOLD $K(\pi,1)$S

Benson Farb mentioned an open question about manifolds recently. Closed odd-dimensional manifolds have Euler characteristic 0. It is conjectured that a closed, aspherical $2n$ manifold M has $(-1)^n\chi(M) \geq 0$. This holds for hyperbolic manifolds, by the Chern-Gauss-Bonnet theorem (and is probably known for homogeneous spaces in general), but is not even known for non-positively curved manifolds. I decided to think about the first non-trivial special cases. It is certainly true in dimension 2, so let's consider dimension 4. Equivalently, we'd like to show that for M^4, if $\chi(M) < 0$, then $\pi_i(M) \neq 0$, for some $i = 2, 3, 4$. We need not consider $|\pi_1(M)| < \infty$, since in this case $\mathrm{sign}\chi(M) = \mathrm{sign}\chi(\tilde{M})$, where $\tilde{M}$ is the universal cover. For simply connected 4-manifolds, $\beta_1 = \beta_3 = 0$, by the Hurewicz theorem and Poincaré duality, so certainly $\chi(M) > 0$ in this case. Another obvious observation is that the conjecture holds for bundles over lower dimensional manifolds. Let's consider the simplest Morse functions. Let $f : M^4 \to \mathbb{R}$ be a self-indexing Morse function with b_i critical points of degree i. We may assume that $b_0 = b_4 = 1$, $b_3 \leq b_1$, and that the critical points of degree i are at height i. Then the condition $\chi(M) < 0$ translates into $b_2 + 2 < b_1 + b_3$. So, in some sense, such manifolds have a "small" number of 2-handles. Thus, let's consider the smallest numbers of 2-handles.

Suppose $b_2 = 0$. Then $b_1 = b_3$, and the manifold is homeomorphic to $(S^1\times S^3)^{\sharp b_1}$. Clearly, $\pi_3(M) \neq 0$. Now, suppose $b_2 = 1$. I mentioned this case to Cameron Gordon, and he came up with the following argument. In this case $b_1 = b_3$ or $b_1 = b_3 + 1$, and $f^{-1}(1.5) = (S^2 \times S^1)^{\sharp b_1}$, $f^{-1}(2.5) = (S^2 \times S^1)^{\sharp b_3}$. This means that as we pass through height 2, a 2-handle addition occurs, and we see a surgery on a knot $k \subset (S^2 \times S^1)^{\sharp b_1}$, after which we obtain $(S^2 \times S^1)^{\sharp b_3}$. Since $b_2 + 2 = 3 < b_1 + b_3 \leq 2b_1$, we must have $2 \leq b_1$. If $N = (S^2 \times S^1)^{\sharp b_1} - \mathcal{N}(k)$ is irreducible, then a theorem of Gabai implies that at most one surgery on k can be reducible. This follows since $rkH_2(N, \partial N) > 0$, so one may find a sutured manifold hierarchy in N which misses ∂N, and with complement a regular neighborhood of ∂N which has finitely many sutures parallel to a slope on ∂N. If one does Dehn filling on any slope other than the suture slope, then the result has a sutured manifold hierarchy, so it is irreducible. Thus, the only possibility in this case is that N is reducible. In fact, by passing to irreducible summands and applying Gabai's theorem, property R, and the knot complement problem, we can show that such examples are obtained by connect sums of $S^1 \times S^3$ and possibly a $\mathbb{CP}^2$. But all that we need to observe is that if N is reducible, there is a non-separating $S^2 \subset N$. This S^2 will be present in $f^{-1}(2)$, since the surgery does not interfere. On either side of $f^{-1}(2)$, it will bound a 3-ball, and putting these together gives a non-separating S^3, so $\pi_3(M) \neq 0$.

These surgery techniques are probably not going to be useful in the general case. When $b_2 = 2$, one can push one critical point below 2, and one above 2, so that 2 is a regular point. Then $f^{-1}(2)$ is a 3-manifold, with two knots in it. Surgery on one knot gives $(S^2 \times S^1)^{\sharp b_1}$ and the other gives $(S^2 \times S^1)^{\sharp b_3}$. So here we are asking when we can obtain the same manifold by surgery on knots in $(S^2 \times S^1)^{\sharp *}$. If there is one such example, then there are probably infinitely many by handle sliding, so the surgery approach seems hopeless.