RESEARCH BLOG 6/8/04
TEXAS TOPOLOGY

Last week, I visited UT Austin. Ken Baker, a student of John Luecke, had his thesis defence on Wednesday. He has a nice result about Berge knots, and his thesis has an impressive number of pictures. These are a special class of knots in S^3 which admit a cyclic surgery. If one considers a genus 2 Heegaard surface S, and a knot $k \subset S$, then one obtains an induced integral framing on k from S. This means that a vector field in S normal to k gives a framing which may wind a number of times around the meridian, but only once around the longitude. If one performs Dehn surgery on k with framing induced by S, then the resulting manifold is obtained by adding a 2-handle to each genus 2 handlebody, and gluing the resulting manifolds together along the new toroidal boundary. Berge realized that if the handle is added along a curve which is not disk-busting in each handlebody (that is, there is an essential disk whose boundary misses k in each handlebody, in which case k is called primitive in the handlebody), then the resulting manifold is a lens space (since handle addition creates two solid tori which are glued together). In this case, k is called doubly primitive. A simple way to obtain doubly primitive knots is to take a fibred genus one knot, and take a simple closed curve lying on the Seifert surface. The genus 2 Heegaard splitting is a regular neighborhood of the punctured torus Seifert surface. If one adds a handle along a push off of a simple closed curve on the punctured torus into the boundary of the regular neighborhood of the punctured torus, then there is an arc disjoint from the curve, and the preimage of this arc under the map which crushes the regular neighborhood back down to a punctured torus gives a disk disjoint from the simple closed curve. Since the knot is fibred, the complementary handlebody also is naturally identified with the boundary of a regular neighborhood of a punctured torus, so the curve is doubly primitive. The only genus one fibred knots are the trefoil (and its mirror image) and the figure eight knot (this follows from a homology computation for surgeries on punctured torus bundles). In fact, Berge has classified all doubly primitive knots. One thing that Ken shows in his thesis is that a Berge knot either comes from surgery on the maximally untwisted five chain link, or is a simple closed curve on a genus one fibred knot. For this class, Ken shows that the volumes of these knots are unbounded. He demonstrates this by showing that one may obtain these curves by arbitrarily high Dehn surgery on hyperbolic links with arbitrarily many components. This means that Berge's knots cannot all be obtained by Dehn surgery on a fixed link. He also gives an efficient algorithm for finding incompressible surfaces in these knot complements, and he shows that infinitely many have no (non-peripheral) closed incompressible surfaces, and infinitely many do. It is conjectured that Berge's knots are the only knots that admit lens space surgeries, but this is still an open conjecture.

This sort of doubly primitive construction generalizes to show that for any genus 1 fibred knot in an exceptional manifold (i.e., not hyperbolike), there will be two exceptional surgeries on a knot lying in the fiber surface. The argument also generalizes to show that knots lying on the twist knot Seifert surfaces (which are also genus 1) have exceptional Seifert fibred surgeries. There are probably other infinite classes of similar examples which admit exceptional surgeries.