RESEARCH BLOG 5/24/04
WHAT NEXT?

At the Cornell Topology Festival, Bill Thurston gave the last talk, entitled "What next?" He initially made some comments about his involvement in 3-manifold topology, and compared himself to Rip Van Winkle, coming back from a mathematical hiatus to find that many of the questions he initiated are claimed to be solved. He commented that many ideas in 3-manifold topology which seemed to lead to divergent areas of study have converged to form intimate relations with each other, such as the study of various topological structures such as laminations, contact structures, geometric structures of various types, and gauge invariants. His main idea for what may be next is to find more relations between these things, and to make more connections with or applications of 3-manifolds to other areas. He then mentioned some questions which he thinks may provide fruitful areas of study.

One is the lamination of all hyperbolic 3-manifolds. The set of all hyperbolic 3-manifolds forms a lamination under the Gromov-Hausdorff (or Chaubauty?) topology: a sequence of points converges if larger and larger balls around the sequence converge in the Gromov-Hausdorff topology. He claimed that this lamination is precompact, but I don't understand why, since there seems to be issues when the injectivity radius approaches zero for a sequence of points. He seems to think this would make an interesting thing to study, but I can't recall any specific questions he asked about it.

Another important question he mentioned is to develop efficient algorithms for classifying 3-manifolds. The geometrization conjecture implies the existence of an algorithm to classify closed 3-manifolds up to homeomorphism. Thurston seems to believe that there may be a polynomial-time algorithm to solve the homeomorphism problem, at least if one already has computed the geometric structure. Thurston suggested that it would be good to find some finite-dimensional model for Ricci flow, such as a piecewise hyperbolic structure. He described hyperbolic structures which are incomplete along a lamination. One way one might approach this is to try to numerically model Ricci flow. If Perelman is correct, then this may converge quite quickly to the geometric structure on the manifold. This seems like quite a challenging problem. Thurston also mentioned algorithms for finding tight contact structures, or other geometric structures on 3-manifolds.

Thurston opened up the floor for other questions. Rich Schwartz mentioned that he thought there may be a connection between the pants complex distance of a Heegaard splitting and the Heegaard Floer homology. I'm not quite sure what he meant by this, but it would be interesting to find conditions on a Heegaard splitting under which one could identify the Heegaard Floer homology as non-vanishing.

Siddartha Gadgil posted a note which gives a new proof of property P making use of Heegaard Floer homology [note added 5/28/04: Gadgil has retracted his paper due to an error]. Gadgil proves a lemma showing that the Heegaard Floer homology invariant of a homotopy S^3 which is ± 1 surgery on a knot in S^3 vanishes. The idea is simple, modulo some claims about the behavior of Heegaard Floer homology which come from papers of Ozsvath-Szabo. Given this, work of Ozsvath-Szabo shows that the reduced Heegaard-Floer homology of ± 1 surgery on a knot in S^3 does not vanish (this makes use of the same ideas going into Kronheimer-Mrowka's proof of property P), which gives a contradiction in conjunction with Gadgil's lemma. I think it should be possible to use Gadgil's method to show that any homotopy 3-sphere has vanishing Heegaard Floer homology. In some sense, this proof is much simpler than that of Kronheimer-Mrowka, since there is much less analysis that goes into the argument, and only one type of Floer homology is used. But Kronheimer-Mrowka's technique has some stronger consequences that may turn out to be more important than the proof of property P.

As mentioned in blog 5/12/04, one corollary of Kronheimer-Mrowka's argument is that the A-polynomial distinguishes knots from the unknot. This is now written up in a paper by Dunfield and Garoufalidis, making use of work of Kronheimer-Mrowka (I gave the wrong citation last blog). One can prove a slightly stronger result from theorem 18 of [1] : if Y is a 3-manifold with a single boundary component, and $H_1(Y) =$

$\mathbb{Z}, H_1(Y(0)) = \mathbb{Z}$ (where $Y(0)$ denotes 0-frame surgery), then all but one Dehn filling on Y has a non-trivial representation to SU(2). This image must be non-cyclic for $Y(1/n)$, thus one concludes that the A-polynomial is non-trivial. In fact, there is a 1-parameter family of representations of $\pi_1(Y)$ into SU(2), which is surprising, since the naive count of expected number of representations is 0-dimensional. The A-polynomial cuts out a curve in $\mathbb{C}^* \times \mathbb{C}^*$, which is a 2-(real)dimensional surface. Representations coming from SU(2) will have eigenvalues lying in the subspace $S^1 \times S^1$. Generically, two surfaces in $\mathbb{C}^* \times \mathbb{C}^*$ will intersect in a finite collection of points, but this result implies that they intersect in a curve. This seems to give a strong restriction on the A-polynomial for such a manifold Y. It would be interesting to know whether this condition holds for all A-polynomials of manifolds with a single torus boundary component?

References

[1] P. B. Kronheimer and T. S. Mrowka. Dehn surgery, the fundamental group and SU(2).