RESEARCH BLOG 3/10/04

Yesterday,Michael Khovanov gave a talk here at UIC. He explained his definition of a graded homology for links $\mathcal{H}^{i,j}$, whose Euler characteristic gives the Jones polynomial. He discovered this by trying to "categorify" the definition of the Jones polynomial via the Kauffman bracket. Since his discovery, he and others have found categorifications of other quantum invariants of knots. Khovanov plotted $\log(rk(\bigoplus_{i,j}\mathcal{H}^{i,j}))$ versus the hyperbolic volume, and notes that the plot looks roughly linear (see his paper). He was inspired to do this by Dunfield's experiment, plotting $\log|J_K(-1)|$ vs. volume. The correlation looks much better in Khovanov's plot, especially for non-alternating knots, but this seems quite mysterious. I speculate that it might also be interesting to do the same plot with the volume of the 2-fold branched cover of the knot instead.

After Khovanov's talk, Paul Seidel described recent work of Jacob Rasmussen. Rasmussen gives a completely combinatorial proof of Milnor's conjecture. The 4-ball genus of a knot $K \subset S^3$ is the minimal genus of a smooth embedded orientable surface $\Sigma \subset B^4$ which bounds $K = \partial\Sigma$. Milnor's conjecture states that the 4-ball genus of a p, q torus knot is $(p-1)(q-1)/2$. This was proven by Kronheimer and Mrowka using Donaldson theory. In fact, Rasmussen shows that for positive knots, the 4-ball genus is equal to the genus in S^3, which for a positive knot is equal to the genus of the Seifert surface of a positive projection (as shown by Stallings). It turns out that the Khovanov homology can be thought of as a sort of relative TQFT of link cobordism. Rasmussen gives a new knot concordance invariant $s(L)$, which is roughly the maximal non-trivial degree in which a version of Khovanov homology is non-trivial. This gives an estimate of the genus of a surface in $S^3 \times [0,1]$ which cobounds the knot in $S^3 \times \{0\}$ and $U \times \{1\}$, where U is the unknot. This is done by showing that it increments byy at most one each time an elementary cobordism (a pair of pants) between a knot and a two component link occurs. His work is based on a refinement of Khovanov homology given by Lee, and by making analogies with a gauge-theory type invariant defined by Ozsvath and Szabo. The advantage of this approach is that it is purely combinatorial, so it avoids a lot of analytic details. It would be quite interesting if one could discover other knot concordance invariants using this machinery.

John Etnyre asked whether given a knot in $\sharp_{2g}\, S^2 \times S^1$ which has a surgery yielding $\Sigma_g \times S^1$, where Σ_g is a surface of genus g, must it be standard, *i.e.* the complement of $D^2 \times S^1 \subset \Sigma_g \times S^1$ (it is a nice exercise to see that surgery along the fiber slope gives $\sharp_{2g}\, S^2 \times S^1$)? This question could be regarded as a generalization of property R, which was proven by Gabai. It seems likely that one could apply Gabai's techniques to answer this question. Etnyre tried this, but got stuck at some point. He asks this question, because it would allow him to show that S^4 admits an achiral Lefschetz fibration. This is essentially a foliation by surfaces, with base S^2, such that there are certain singular fibers over isolated points of S^2. A 4-manifold admits a chiral Lefschetz fibration (Lefschetz pencil) if and only if it is symplectic (this was shown by Gompf and Donaldson). It is unclear whether achiral Lefschetz fibrations have any topological significance. This seems analogous to open book decompositions on 3-manifolds: these are equivalent to contact structures and any 3-manifold has one, but only certain ones correspond to tight contact structures.