RESEARCH BLOG 11/24/03

I've been working on making the argument sketched in blog 11/16/03 more precise. What we need to show is that for a geometrically finite end of a complete pinched negative curvature Riemannian (PNC) manifold M (with sectional curvature satisfying $-a^2 \leq K_{sec} \leq -1$, $a > 1$), the boundary of the convex core $C(M)$ has bounded area, only depending on the pinching constants and the topological type of the boundary. This follows from the following argument of Kleiner [3]. Let E_s be the points in M distance $\leq s$ away from $C(M)$, so $E_0 = C(M)$. The surfaces ∂E_s are $C^{1,1}$ for $s > 0$, so they are C^2 almost everywhere, by Rademacher's theorem. Then $\int_{\partial E_s} GK_s da \to 0$ as $s \to 0^+$ (he was taking the convex hull of a compact set, but the argument is local, so it generalizes to convex hull of limit sets), where GK_s is the Gauss-Kronecker curvature of ∂E_s, the product of the principal curvatures. One can thus make sense of the equation $K = K_{sec} + GK_s$ at points where ∂E_s is C^2, where K is the intrinsic sectional curvature of ∂E_s, and K_{sec} is the sectional curvature of the plane tangent to ∂E_s. Since $K_{sec} \leq -1$, we have

$$2\pi\chi(\partial E_s) = \lim_{s\to 0^+} \int_{\partial E_s} K da = \lim_{s\to 0^+} \int_{\partial E_s} K_{sec} + GK_s da$$

$$\leq \lim_{s\to 0^+} \int_{\partial E_s} -1 da = -\lim_{s\to 0^+} Area(\partial E_s),$$

thus $Area(\partial E) \leq -2\pi\chi(\partial C(M))$.

Once we have a bound on the area of $\partial C(M)$, we may get a bound on the shortest length geodesic, using a systolic inequality. Let $sys(\Sigma)$ be the shortest length homotopically non-trivial geodesic on the surface Σ (equipped with a Riemannian metric). If Σ is compact, then Gromov proves

$$\frac{1}{2} sys(\Sigma)^2 \leq Area(\Sigma).$$

[2] Gromov's argument generalizes to surfaces with $genus > 1$, but unfortunately I wasn't able to generalize the argument to planar surfaces. But since our goal is to find systoles on $\partial C(M)$, we can use some special properties of these surfaces to get a systole inequality. In the situation at hand, I may assume that the cusps of M are standard, that is they have a neighborhood H isometric to the quotient of a horoball in $\mathbb{H}^3_{-b^2}$, for some $b > 0$. The $\partial C(M) \cap H$ will be a pleated surface, so has an intrinsic metric of sectional curvature $-b^2$. Moreover, we may find horocycles embedded in $\partial C(M)$ of length the same as the length of a homotopic curve in ∂H, say l (here, one needs to assume that $\partial C(M)$ is π_1-injective, using an argument of [1]). Thus, we may truncate $\partial C(M)$ to a surface Σ having a boundary component of length $\geq l$. Now, we may apply the Besicovitch lemma, which implies that the area of Σ is $\geq 2l' inj(\partial\Sigma)$, where l' is the shortest length curve homologous to the boundary, and $inj(\partial\Sigma)$ is the injectivity radius of the boundary, that is $\frac{1}{2}$ the shortest length homotopically non-trivial curve joining the boundary to itself. If $l' \geq l$, then $inj(\partial\Sigma)$ is bounded by $Area(\Sigma)$, and we may surger the boundary along a curve of length $2inj(\partial\Sigma)$ to get a geodesic of bounded length. Otherwise $l' < l$ and we can find a geodesic in Σ of length $< l$.

Thus, we can bound the systole of $\partial C(M)$. We can use this to show that $\partial C(M)$ is either bounded distance to the (compact components of the) thin part of M, or $\partial C(M)$ is of bounded distance to a simplicial ruled surface.

REFERENCES

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