RESEARCH BLOG 11/12/03

Last week, I decided to take a break from thinking about Ricci flow. I was working hard on computing the variation of the Willmore energy under Ricci flow, and I came up with a computation, but I couldn't see how to make use of it as a maximum principle, to show that a Willmore energy minimizer is increasing with Ricci flow (as conjectured in blog 10/17/03). Also, I wasn't able to prove the conjecture on backwards propagation of minimal surfaces under Ricci flow, as conjectured in blog 3/4/03. The main reason is that for an unstable minimal surface Σ, $\partial^2 A(\Sigma)/\partial r^2$ could be > 0 (that is, the second variation of area in the normal direction may be positive, even though there is *some* variation which has negative 2nd variation), so I wasn't able to make the heuristic I gave precise. Right now, I don't have any ideas on how to approach these conjectures.

In blog 10/28/03, I worked out an improvement of the Bishop-Gromov comparison theorem in the case of negatively curved 3-manifolds. I stated that I thought the inequality should hold more generally. But the difficulty is that the exponential map may become orientation reversing when there are conjugate points, such as for S^3 with the standard metric. Thus, one can underestimate the volume of a ball of radius r by estimating the 2nd derivative of spheres and integrating. In Bishop's theorem, he gets around this by stopping the comparison at the cut locus, but I can't do this in my approach, since I need to use Gauss-Bonnet to integrate over the whole sphere. Thus, I don't know if the inequality holds for general metrics on 3-manifolds.

I took a look at the paper of Cao and Meyerhoff, proving that the minimal volume orientable cusped hyperbolic 3-manifolds are the figure eight knot complement and its sibling [4]. In this paper, they prove that the boundary of a maximal cusp always has area > 3.35. I used this result in a paper to show that the maximal number of exceptional surgeries on a cusped hyperbolic 3-manifold is ≥ 12. Unfortunately, Cao and Meyerhoff use rigorous computation to establish their results, although they state in their paper that one could probably do it by hand. Last week, I worked out how to do this for part of their proof, and found a trick which simplifies the argument (another part was worked out by Colin Adams and some students using computation, who can prove that a maximal cusp has volume $\geq \sqrt{3}$, which is enough to establish the minimal volume result by a packing argument of Boroczky, which is what I can also do by hand). The argument comes down to minimizing a certain function over a 4-dimensional parameter space. One can analyze how the function varies when one changes one parameter at a time, and find the point that minimizes the function directly. The reason I'm interested in this is that is that if one could improve lower bounds on volumes of cusped manifolds, one gets improved lower bounds on closed manifolds by the drilling correspondence I used in [1].

Last Friday, I started thinking about Marden's conjecture again. This conjecture (stated as a question by Al Marden) states that a finitely generated Kleinian group Γ (discrete subgroup of $PSL_2(\mathbb{C})$) has tame quotient, i.e. $\mathbb{H}^3/\Gamma = int(M)$, where M is a compact 3-manifold. My advisor, Mike Freedman, worked on this when I was a graduate student. At the time, I wasn't so interested in the question, since I knew very little about hyperbolic 3-manifolds or Kleinian groups, and the problem seemed quite intimidating. But I've run up against the Marden conjecture in my research several times.

The Marden conjecture would complete the classification of Kleinian groups. I worked out a classification of non-free 2-parabolic generator Kleinian groups. For the free case, the groups should lie in the closure of the Riley slice of Schottky space. With the resolution of the ending lamination conjecture, the Marden conjecture would finish the classification of 2-parabolic generator groups. Another major corollary of Marden's conjecture is that for a finite volume hyperbolic 3-manifold M, and cover with finitely generated fundamental group is either geometrically finite, or is a geometrically infinite surface group which is a virtual fiber (this was proven by Canary [3]). Darren Long, Alan Reid, and I proved that the Bianchi groups are subgroup separable on geometrically finite subgroups (Dani Wise proved the same result for the fundamental groups of several alternating knot complements). Thus, Marden's conjecture would imply that the Bianchi groups are LERF ("Locally Extended Residual Finite"), which just means that every finitely generated subgroup is separable, i.e. the intersection of all finite index subgroups containing it. Other applications of Marden's conjecture would be to obtain improved Margulis constants of free groups and volume estimates of hyperbolic manifolds, via theorems of Culler, Shalen, and their collaborators [5, 2]. Their results would also enable me to prove that there are only finitely many one-cusped hyperbolic 3-manifolds which have more than 8 exceptional Dehn fillings.

The approach I'm considering is inspired by ideas of Canary and Freedman. The approach makes use of results of Souto described in blog 3/7/03 and ideas of Myers mentioned in a talk from the Spring Topology and Dynamics conference earlier this year. I'll attempt to describe them in a later blog.

References

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[3] R. D. Canary. A covering theorem for hyperbolic 3-manifolds and its applications. *Topology*, 35(3):751–778, 1996.

[4] C. Cao and G. R. Meyerhoff. The orientable cusped hyperbolic 3-manifolds of minimum volume. *Invent. Math.*, 146(3):451–478, 2001.

[5] M. Culler and P. B. Shalen. Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 5(2):231–288, 1992.