

RESEARCH BLOG 10/31/03

In the last research blog (10/28/03) , I gave a strengthening of the Bishop-Gromov comparison theorem in dimension 3 using the pinching assumption $Rg - Ric \geq -4g$. The exact same argument shows that with the same pinching estimates, the volume of a tube about a geodesic is less than the volume of a tube of the same radius about a geodesic of the same length in \mathbb{H}^3 . Usually, such estimates involve lower bounds on sectional curvature. I considered whether this theorem can be generalized to higher dimensions. The difficulty seems to be that $\int_{S^n} R(g)dV(g)$ with $\int_{S^n} dV(g) = 1$ might be unbounded for $n > 2$, so one cannot control this term of the 2nd variation formula for spheres. I do not know a reference for this, though. It would be interesting to know if volume entropy (normalized so the volume is 1) is increasing under Ricci flow on 3-manifolds, but this seems like a difficult computation. I can check directly that the volume entropy remains bounded under Ricci flow (making use of Hamilton's pinching estimate for the curvature in 3-D and my entropy estimate), independent of Perelman's claim of geometrization.

I claimed in blog 2/28/03 that I could not verify Hamilton's formula

$$“\Delta”g = -2Ric,$$

where “ Δ ” is the Euclidean Laplacian in normal coordinates. I think I've figured out what was wrong with my objection (I probably made a computational error as well). The expansion for an analytic metric in normal coordinates about a point m is given by

$$g_{pq} = \delta_{pq} - \frac{1}{3} \sum_{ij=1}^n R_{ipjq}(m)x_i x_j + \dots$$

(see Cor. 9.8, p. 209, [1]), where the coefficients of the Taylor expansion depend on derivatives of the curvature tensor at m . Then since $R_{iiii}(m) = 0$, this means $\frac{\partial^2}{\partial x_i^2} g_{ii} = 0$, so that what I claimed was unequal

is actually an equality. To verify Hamilton's claim, notice

$$“\Delta” g_{pq} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} g_{pq} = -\frac{1}{3} \sum_{i=1}^n R_{ipiq} \frac{\partial^2}{\partial x_i^2} x_i^2 = -\frac{2}{3} R_{pq}(m).$$

This justifies Hamilton's analogy of Ricci flow with the heat equation.

REFERENCES

- [1] A. Gray. *Tubes*. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1990.