RESEARCH BLOG 10/17/03

Kronheimer, Mrowka, Ozsvath, and Szabo posted a paper on Monday, which proves many results about Dehn surgeries on knots (although the results depend on a paper of Kronheimer and Mrowka which is in preparation). In particular, it gives a new proof of the knot complement problem, which is independent of Gordan and Luecke's proof. The paper draws on many deep results and techniques of 3- and 4-manifold topology from the past couple of decades. When I was a graduate student, I had heard from Mike Freedman that Kronheimer and Mrowka were putting the finishing touches on a proof of property P (which is the conjecture that no non-trivial Dehn filling on a knot complement in S^3 can give a homotopy sphere, which would be a consequence of Poincaré conjecture). I believe this paper is the write-up of that proof, although they do not get homotopy information, possibly because they haven't proven homotopy invariance of the monopole invariants.

Let's investigate why 4-manifold topologists are interested in surgery on 3-manifolds. A smooth 4-manifold has a handle decomposition, coming for example from a Morse function, with handles of five possible indices. It is natural to investigate 4-manifolds which may be constructed from the fewest number of handles. There must always be at most one 0-handle, and one 4-handle, both of which are balls. Thus, the only 4-manifold with two handles is S^4. If a 4-manifold has 3 handles, then it must have handles of indices 0, 2, and 4, by Poincaré duality, and each of these handles generates non-trivial homology in the corresponding dimension. One can build up the manifold by seeing how the topology of the level sets of the Morse function changes as one passes through the critical points. At the start, one has a 0-handle, which is a 4-ball. The level sets near the index 0 critical point are S^3. By flipping the manifold over (negating the Morse function), we see that the level sets near the index 4 critical point are also S^3. When one passes through the index 2 critical point, a surgery is performed on the level sets. One attaches a handle of the form $D^2 \times D^2$, which has boundary $(S^1 \times D^2) \cup (D^2 \times S^1)$. Thus, passing through the critical point, one glues $S^1 \times D^2$ to S^3. The core of the attaching region is a knot in S^3, and the D^2 factor represents a normal disk bundle to the knot, giving a regular neighborhood of the knot. The attaching map is given by a diffeomorphism of $S^1 \times D^2$ up to isotopy. There are a $\mathbb{Z}$'s worth of such diffeomorphisms, given by cutting along a D^2 and twisting. These correspond to $\mathbb{Z}$-framed Dehn fillings on the knot, given by how many times one twists the D^2 (and which direction). In our situation, we perform integral Dehn surgery and get back S^3. The only Dehn fillings on a knot which yield a homology sphere are of the form $1/n, n \in \mathbb{Z}$ (the Dehn filling p/q is performed by attaching a solid torus with meridian $p\mu + q\lambda$, where μ is the meridian and λ is the longitude). Thus, the surgery must have framing ± 1. Now, by the solution of the knot complement problem by Gordon and Luecke, the only Dehn filling of a knot which gives back S^3 is ∞ surgery, unless we have the unknot. For the unknot, $1/n$ filling gives back S^3. So we must have done ± 1 surgery on the unknot. The corresponding 4-manifolds are homeomorphic (by an orientation reversing homeomorphism), and are homeomorphic to $\mathbb{CP}^2$. Thus, there is essentially one manifold with exactly 3-handles. Kronheimer et al. are claiming to have reproved this theorem.

What about 4-manifolds with four handles in a handle decomposition? There are several possible indices of the handles here (again, appealing to Poincaré inequalities, up to flipping the manifold over): $(0,0,1,4), (0,1,2,4), (0,1,3,4), (0,2,2,4)$. The first case must be S^4. A movie of the manifold would be two balls, with a 1-handle attached, then a ball to cap it off. To see how the level sets change attaching a 1-handle, one removes two balls from the 3-manifold, then glues in $S^2 \times I$ to the resulting boundary. If the balls lie in the same component, then we get a connect sum with $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$, otherwise we just get a connect sum of the corresponding components. Since we must get S^3 level set after passing through the 1-handle, we see that the 1-handle performs connect sum on the two S^3 level sets. Cancelling a 0- and 1-handle gives shows that we may reduce to two handles, and have S^4.

The second case with indices $(0,1,2,4)$ can be analyzed as follows. After the index 1 critical point, the level sets are $S^2\times S^1$ (for orientation reasons), and between indices 2 and 4, the level sets are S^3 (surgeries on knots in S^3 are always orientable since $D^2\times S^1$ is orientable). So we see $S^2 \times S^1$ obtained by surgery on a knot in S^3. For homological reasons, the surgery has framing 0, that is the meridian of the solid torus is glued to the longitude of the knot. So to classify such 4-manifolds, one needs to know which knots in S^3 give $S^2 \times S^1$ when one performs 0-framed surgery. Dave Gabai solved this problem, and proved something much stronger, that the resulting manifold has a taut finite depth foliation which is transverse to the core of the Dehn filling, containing a depth 0 leaf which is obtained by capping off a minimal genus Seifert surface for the knot. Since the only taut foliation of $S^2 \times S^1$ is the standard foliation by spheres, and the only knot transverse to this foliation (going around it once, for homological reasons) is an S^1 factor, we see that the knot is the unknot. Thus, the only such manifold has level sets obtained by performing 0-framed surgery on the unknot. The manifold then is S^4, since the 2-handle can be cancelled geometrically with the 1-handle.

The third case with indices $(0,1,3,4)$ corresponds to taking two copies of $B^3 \times S^1$ or $B^3\tilde{\times}S^1$, and gluing along the boundaries $S^2 \times S^1$. Since S^2 factors must correspond to S^2 factors in this gluing (up to isotopy), we see that the resulting manifold is $S^3 \times S^1$ or $S^3\tilde{\times}S^1$.

The fourth case of indices $(0,2,2,4)$ corresponds to performing surgery on 2-component links in S^3 and obtaining S^3 back. An example is $S^2\times S^2$. To classify such links is a difficult question, and I believe there are non-trivial examples, such as the Berge link. Since these manifolds are simply connected, they are classified up to homeomorphism by a result of Freedman, which says that they are essentially classified by their cohomology ring, plus the Kirby-Seibenmann invariant. But as far as I know, this doesn't give a classification of the possible surgery link types.

Now I'll outline the argument of Kronheimer et al., although I don't understand most of the details (and I might get part of the outline incorrect). Kronheimer and Mrowka define a monopole Floer homology, which "counts" SU(2) connections in a special way. These invariants are related to Seiberg-Witten invariants of 4-manifolds with boundary. A theorem of Taubes shows that the Seiberg-Witten invariants of a symplectic 4-manifold correspond to invariants of Gromov which count pseudo-holomorphic curves. A 3-manifold with a contact structure is symplectically fillable if it bounds a symplectic 4-manifold such that the symplectic form induces the contact structure on the boundary. I believe that Kronheimer and Mrowka show that their invariants are non-trivial on such symplectically fillable manifolds, due to Taubes' work. The monopole invariants obey the Floer exact triangle, which given a knot, gives an exact triangle of the Monopoloe Floer homology for the 0-framed surgery, ∞-surgery, and 1-surgery. By Gabai's result, the 0-surgery has a taut foliation if the knot is non-trivial. Eliashberg and Thurston show that one may deform the tangent planes to this foliation to get a symplectically fillable contact structure. Thus, the monopole Floer homology is non-trivial. Also, the ∞ surgery has trivial Floer homology. Thus, by exactness of the Floer triangle, the 1-framed surgery must have non-trivial Floer homology, and thus cannot be S^3. They use similar sorts of arguments to answer a question of Gordon, to show that the only way to obtain $\mathbb{RP}^3$ by surgery on a knot is ± 2 surgery on the unknot. They also apply their techniques to study lens space surgeries on knots, and obstructions to taut foliations. For example, they claim that the 2-fold branched cover of an alternating knot has no taut foliation. This is a remarkable result, since previous proofs that manifolds contained no taut foliation used detailed case-by-case analysis.

I've been working on conjectures I made in blog 10/6/03 . I have a modified conjecture, although this is probably wishful thinking. Given an immersed surface $\Sigma \to M^3$, M^3 a Riemannian 3-manifold, the *Willmore energy* is $\int_\Sigma |H^2| da$, where H is the mean curvature. Of course, one may define this for hypersurfaces in any dimension, but in dimension 3, this energy is scale invariant (and in fact is Möbius invariant for $\mathbb{R}^3$). What I conjecture is that the minimal Willmore energy of 2-spheres which are very close to being geodesic increases under the Ricci flow. This conjecture was verified for warped products. By Perelman's work, when a singularity occurs in the Ricci flow, then there are 2-spheres close to being geodesic with arbitrarily small Willmore energy, since rescaled limits approach $S^2 \times \mathbb{R}$. Thus, the conjecture would imply that there exist 2-spheres with arbitrarily small Willmore energy which are close to geodesic at times near to the singularity time. Presumably then, this would imply that there exist spheres with zero Willmore energy, which are therefore minimal. I'm attempting to compute the evolution of Willmore energy of a surface under Ricci flow, but the computation is quite involved.

In blog 8/20/03, I described a comparison formula for volumes of hyperbolic manifolds with metrics with pinched scalar curvature, which would follow from Perelman's claims. I claimed that this would give an improvement on the smallest volume hyperbolic 3-manifold. Last week, Nathan Dunfield wrote me to explain how to improve this estimate slightly, by improving a metric construction, and I improved a bit on this. Now the estimate we get for the minimal volume hyperbolic 3-manifold is .649, which is about a 2-fold improvement on previous estimates using BCG and Ricci pinching. We probably won't publish this result until Perelman's work has been checked in detail, but I'll try to describe this construction in a later blog, as well as give some data that Dunfield obtained to test Perelman's estimates.