

Perelman states in his second paper that one of his goals is to make a canonical Ricci flow, which exists on a maximal subset of space time. One possible application of this would be to the Smale conjecture (and its generalizations), although it should also follow if one can perform the surgeries continuously. For the S^3 , the Smale conjecture states that the space of diffeomorphisms is homotopy equivalent to $O(4)$, the group of isometries of the round 3-sphere. This was proven by Hatcher [2]. Similarly, for a hyperbolic manifold, the group of diffeomorphisms is homotopy equivalent to the isometry group. This was proven by Gabai [1]. There are similar statements for other manifolds satisfying the geometrization conjecture, most of which have been solved. To try to prove this using Ricci flow, one could use Gabai's strategy. It is equivalent to show that all the homotopy groups of $Diff(M)$ are trivial, since it is known that $Diff(M)$ is homotopy equivalent to a cell complex. Gabai starts with a hyperbolic manifold (M, g) , and considers an S^n family of diffeomorphisms, $f : S^n \rightarrow Diff(M)$. Then $f^*(g)$ is an S^n family of hyperbolic metrics on M , $f^*(g) : S^n \rightarrow Riem(M)$. Since $Riem(M)$ is convex, and therefore contractible, one can extend this to a balls worth of metrics, $F : B^{n+1} \rightarrow Riem(M)$, where $F|_{S^n} = f^*(g)$. Then Gabai uses this to find a B^{n+1} 's worth of hyperbolic metrics. One could instead try to use Ricci flow. Let's consider this in the S^3 case. One would perform the "canonical" Ricci flow, and for each metric $F(g)$, the flow would converge to a round metric in finite time, going "extinct". Now, modify F so that all the extinction times are the same. The Ricci flow is fixed on the boundary, up to scaling. If the canonical Ricci flow exists, and varies continuously enough with respect to the initial metrics, then there should be some time after which no singularities form before the extinction. Then the metrics would converge to a ball's worth of round metrics. There are some difficulties here, since solutions may go extinct without becoming round in the case of type II singularities (see DG/0306129 for evidence of this),

so maybe one would want to perform surgeries continuously, rather than working with a canonical flow. In the hyperbolic case, it would probably be useful to know that only finitely many surgeries occur.

REFERENCES

- [1] D. Gabai. The Smale conjecture for hyperbolic 3-manifolds: $\text{Isom}(M^3) \simeq \text{Diff}(M^3)$. *J. Differential Geom.*, 58(1):113–149, 2001.
- [2] A. E. Hatcher. A proof of a Smale conjecture, $\text{Diff}(S^3) \simeq \text{O}(4)$. *Ann. of Math. (2)*, 117(3):553–607, 1983.