Colding and Minicozzi posted a paper on singularity formation in the Ricci flow (thanks to Mohan Ramachandran for pointing this out to me). They prove that if one has a 3-manifold M which is non-aspherical, then Ricci flow can exist for only finite time. This is stronger than what Perelman proved in his third paper, in which he proved this for manifolds which have virtually free fundamental group (although Perelman claims a stronger result, that the Ricci flow with δ-cutoff becomes a connect sum of spherical space forms in finite time in this case). Colding and Minicozzi prove this using similar methods to Perelman, although they are technically somewhat simpler. If $\pi_3(M) \neq 0$, then they use area estimate of a minimax sphere to show that the solution must have a singularity in finite time, as described in blog 7/20/03. If $\pi_2(M) \neq 0$, then they give a more complicated estimate of the change in area of a minimal 2-sphere which is homotopically non-trivial, and again show that this leads to a singularity in finite time. Presumably their technique would generalize to Ricci-flow with δ-cutoff, but I don't understand this well enough yet.

In blog 7/20/03, the monotonicty formula for $\mu(g, \tau)$ I stated at the end was incorrect, actually one obtains the same monotonicity formula as Perelman, $\mu(g_0, \tau) \leq \mu(g_t, \tau - t)$ for solutions to the Ricci flow $g_t, 0 \leq t \leq \tau$. Since the gradient flow as given is scaled Ricci flow, one must scale space and time to obtain the monotonicity formula, which I failed to do correctly at first.

I think that Perelman's work gives a new approach to proving that Ricci flow on surfaces uniformizes the surface. Hamilton proved in [4] that if one has a closed surface S with either $\chi(S) \leq 0$, or $\chi(S) > 0$ and the curvature is positive, then the Ricci flow converges to the constant curvature metric. Ben Chow [3] finished the case where $\chi(S) > 0$, but the metric might be 0 somewhere. In Hamilton's paper, he proves that the only soliton on $\mathbb{R}^2$ is the cigar soliton Σ, a metric given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

This metric is asymptotic to a cylinder of radius 1 as $r \to \infty$, and the curvature is asymptotic to e^{-r} (where r denotes distance to the origin). This metric is κ-collapsed at all scales, and thus by Perelman's work may not occur as the limit soliton of a metric (see blog 7/23/03 for an explanation). To see this, note that as $r \to \infty$, the ball of radius r far out enough in the soliton has area $\sim Cr$, since it is asymptotic to a cylinder, but the curvature approaches 0, so it is κ-collapsed, for any κ. Thus, the only singularity which may occur is a gradient shrinking soliton, which must be the round 2-sphere (see theorem 26.1 of [5]). I believe that this gives an alternative to Chow's argument, but I haven't checked things in detail yet (I'm sure that the experts could explain this to me immediately). In fact, the same argument shows that $\Sigma \times S^1$ and $\Sigma \times \mathbb{R}$ cannot occur as limit solitons of the Ricci flow in 3-dimensions, since they are κ-collapsed at all scales. Thus, a solution to the Ricci flow cannot "pinch" along a knot in finite time (from the classification of limit solutions, again see [5], section 26, theorem 26.5). This is one of the significant contributions of Perelman's work to Hamilton's program.

In section 8 of Perelman's second paper, he claims that one can apply a technique of Anderson to show that the limiting manifold of Ricci flow with δ-cutoff gives hyperbolic pieces whose torus boundary components are incompressible. I'll briefly describe what I think he means here. For a metric g on M, let $\lambda(g)$ denote the lowest eigenvalue of $-4\Delta + R$, and let $\overline{\lambda}(g)$ denote $\sup_g \lambda(g) V^{2/3}$, where V denotes the volume with respect to the metric g. If $\lambda(g) < 0$, then $\lambda V^{2/3}$ is increasing along Ricci flow. For a hyperbolic manifold (M, g) of sectional curvature -1, $\lambda(g) = -6$, so $\lambda(g) V^{2/3} = -6V^{2/3}$. For a manifold M, consider $\nu(M) = (-\frac{1}{6}\overline{\lambda})^{3/2}$. Since the volumes of hyperbolic 3-manifolds are well-ordered, one may choose a metric g on M such that there is no hyperbolic manifold N satisfying $\nu(M) \leq Vol(N) < \nu(g) = (-\frac{1}{6}\lambda(g))^{3/2} V$, and use g to run Ricci flow with δ-cutoff. Perelman claims that in the limit as $t \to \infty$

of the Ricci flow, the metric converges to hyperbolic pieces and graph manifold pieces. But a priori, one of the graph manifold pieces could be a solid torus, which performs Dehn filling on a cusp of a finite volume hyperbolic piece N giving a piece $N(\alpha)$. Presumably, Perelman also shows that λ converges in the limit, so that $\nu(M)$ is the sum of volumes of the hyperbolic pieces of the limit. Now, an argument of Anderson (see Perelman's paper for the reference) shows that one may insert a metric on $N(\alpha)$ such that the scalar curvature is ≥ -6 (so $\lambda(g) \geq -6$, since -4Δ is a positive operator), and the volume decreases. Thus, $\nu(M) \leq \nu(N(\alpha)) < \nu(N) < \nu(g)$, which gives a contradiction to the choice of g, since one must then have $\nu(N) = vol(N) = \nu(M) > \nu(N(\alpha))$.

One corollary of Perelman's monotonicity formula for $\overline{\lambda}$ is a strong volume comparison formula. Given a hyperbolic manifold N, if one puts a metric g on N with $R(g) \geq -6$, then $\mathrm{Vol}(N, g) \geq \mathrm{Vol}(N)$. This is much stronger than a result of BCG [2], which implies the same conclusion under the assumption that $Ric(g) \geq -2g$ (equality holds in the hyperbolic case). Anyway, one can apply this to get improved estimates on the minimal volume hyperbolic 3-manifold, using the technique of [1]. A back of the envelope computation shows that the minimal volume hyperbolic manifold has volume $\geq .547$, but I haven't checked this computation carefully yet, although it is clear that there is some improvement on the previous estimates (the result depends on computer aided proofs). It is conjectured that the Weeks manifold, of volume .9427 is the minimal volume, so this is still a ways off.

I have stated some conjectures on minimal surfaces and Ricci flow. I believe that if these conjectures hold, then given a hyperbolic manifold N, the hyperbolic metric minimizes the "number" of minimal surfaces over any metric on N. This would be a surprising result, and might make a good test for whether the conjectures are plausible. For example, if one has a hyperbolic manifold N which fibers over S^1, there is always a stable minimal surface. If one takes a finite cyclic cover $\tilde{N}$ (dual to the fiber) of order n, then there will be n disjoint parallel stable minimal surfaces which are copies of the fiber. Then any metric on $\tilde{N}$ would have to have at least n stable minimal surfaces which are copies of the fiber. This seems quite implausible, but I haven't been able to construct a counterexample (one possible counterexample would be to take a Sullivan metric in which it is foliated by minimal leaves, and then scale the leaves so that there is one minimal and one maximal area leaf. Then I would expect that there are no other minimal surface copies of the fiber, but have been unable to prove this). So this makes me suspicious of my conjectures, although I still think that the heuristics I gave could be made rigorous.

Given a manifold, what is the maximizer of $\lambda V^{2/3}$ in a conformal class of metrics? I believe that it should be a Yamabe metric, and that the gradient flow should give flow by scalar curvature, which I think has been shown to converge to the Yamabe metric. This indicates that one might be able to use Ricci flow to complete Anderson's approach to geometrization by maximizing scalar curvature of Yamabe metrics, and also that one might be able to use Perelman's techniques to understand the scalar curvature flow.

Another thing I would like to understand is limits of κ-non-collapsed metrics with no bounds on curvature (except at the origin). Such limits may be useful in understanding type IIa singularities in Ricci flow, but I'll have to understand Hamilton's compactness arguments before trying to work on this. The idea would be to try to get a limit of blow-ups near a singularity to an incomplete soliton which might have more information than the normal type of limits.

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