RESEARCH BLOG 7/20/03

On Friday, Perelman posted a new paper, which claims to finish off the "elliptization" conjecture, along with the results from [2] and [1]. He proves that if a closed Riemannian 3-manifold M has a prime decomposition which contains no aspherical factors, then the Ricci flow with cutoff on M, defined in [1], will become extinct in finite time. If M has a prime decomposition with no aspherical factors, then M is a connect sum of copies of $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, and manifolds with finite fundamental group. It is well known that $S^2 \times S^1$ and $S^2 \tilde{\times} S^1$ are the only prime 3-manifolds with $\pi_2(M) \neq 0$. Also, $|\pi_1(M)| < \infty$ if and only if there is a non-zero degree map $f : S^3 \to M$, so $\pi_3(M) \neq 0$ if $|\pi_1(M)| < \infty$. Conversely, one may show that if $\pi_3(M) \neq 0$, then either $\pi_2(M) \neq 0$, in which case there is an embedded non-trivial S^2 by the sphere theorem, or $|\pi_1(M)| < \infty$. The idea is to lift a non-trivial map $S^3 \to M$ to $\tilde{M}$, the universal cover. If $|\pi_1(M)| = \infty$ and $\pi_2(M) = 0$, then $\tilde{M}$ is non-compact, and we may engulf the image of S^3 by a handlebody, in which case we see that the map is homotopically trivial. If $\pi_2(M) = \pi_3(M) = 0$, then M is aspherical, by the Hurewicz theorem. Putting these facts together shows the above claim. An equivalent characterization is that $\pi_1(M)$ is virtually free.

Perelman's idea is to find a non-trivial S^2's worth of contractible loops in M, and consider the maximal minimal area disk bounded by these loops in a given metric, then takes the infimum over all such classes. Perelman shows that this value goes to zero in finite time, for solutions to the Ricci flow with cutoff, and thus the solution must go extinct in finite time (by the assumption, surgeries on 2-spheres keep the manifold aspherical). The loops act as a sort of noose, cinching down the 3-manifold to collapse in finite time. This result is somewhat technical, but Perelman gives a good intuitive argument as to why this should work in the first section of his paper.

I believe another approach to this would use a mini-max 2-sphere. If one considers sweepouts by 2-spheres of non-zero degree, and takes a sweepout of minimal maximum area A, then this should satisfy the equation $\frac{dA}{dt} = \frac{\partial^2 A}{\partial r^2} - 8\pi$ (see blog 5/21/03). Here, r is the parameter of a normal variation to the maximal sphere, and by the maximality $\frac{\partial^2 A}{\partial r^2} \leq 0$. So A should go to zero in finite time. One would need to check though that A is non-increasing as one performs a surgery, but I don't understand this part of Perelman's argument in [1] to see if this is the case. Also, establishing the existence of minimax 2-spheres is probably rather tricky, see Colding-De Lellis.

I need to correct some things from the end of blog 7/15/03. In section 3 of Perelman's paper [2], he defines a functional

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(|\nabla f|^2 + R) + f](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV - n,$$

for f satisfying $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1$, $\tau > 0$. Letting $u = f + \frac{n}{2}\ln(4\pi\tau)$, we see that $\int_M e^{-u} dV = 1$ and

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(|\nabla u|^2 + R) + u] e^{-u} dV - n - \frac{n}{2}\ln(4\pi\tau)$$
$$= \tau\mathcal{F}(g, u) - n - \frac{n}{2}\ln(4\pi\tau) + \int_M u e^{-u} dV.$$

If we choose u such that $\mathcal{F}(g, u)$ is minimal, then $\mathcal{F}(g, u) = \lambda(g)$, the minimal eigenvalue of $-4\Delta + R$, as described last time. Thus, we see that

$$\mathcal{W}(g, u - \frac{n}{2}\ln(4\pi\tau), \tau) = \tau\lambda - n - \frac{n}{2}\ln(4\pi\tau) + \int_M u e^{-u} dV.$$

Letting $\mu(g, \tau) = \inf \mathcal{W}(g, f, \tau)$, we see that

$$\mu(g, \tau) \leq \tau\lambda - n - \frac{n}{2}\ln(4\pi\tau) + \int_M u e^{-u} dV.$$

If $\lambda \leq 0$, then we see that $\nu(g) = \inf \mu(g, \tau) = -\infty$, since $\lim_{\tau\to\infty} \mu(g, \tau) = -\infty$. But Perelman claims that $-\infty < \nu(g) < 0$ if $\lambda > 0$. This follows from $\lim_{\tau\to\infty} \mu(g, \tau) = \infty$ if $\lambda(g) > 0$, and from Claim 3.1. Choose u such that $\Phi = e^{-u/2}$, $\int_M \Phi^2 dV = 1$, and $\tau(-4\Delta\Phi + R\Phi) - 2\Phi\ln\Phi = \mu\Phi$. Then $\mu(g, \tau) = \int_M [\tau(-4\Delta\Phi + R\Phi) - 2\Phi\ln\Phi]\Phi dV - n - \frac{n}{2}\ln(4\pi\tau) \geq (\tau - 1)\lambda + \mu(g, 1) - \frac{n}{2}\ln(\tau)$, thus $\lim_{\tau\to\infty} \mu(g, \tau) = \infty$ if $\lambda(g) > 0$. So $\nu(g)$ is a scale-invariant quantity which is monotonic with respect to the Ricci flow, but it is only interesting when $\lambda > 0$. I haven't figured out how to make Claim 3.1 rigorous yet.

A similar argument to that made in blog 7/15/03 shows that

$$\delta\mu(g,\tau) = \int_M \frac{\tau}{2} e^{-u}[v_{ij}(-2(R_{ij} + \nabla_i\nabla_j u) + \frac{1}{\tau} g_i j] dV,$$

where u is the normalized minimizer of $\mathcal{W}(g, u - \frac{n}{2}\ln(4\pi\tau), \tau)$, and we keep τ fixed. Thus, the flow $g_t = -2(Ric + \nabla^2 u) + \frac{1}{\tau} g$ is a gradient flow with respect to the metric $\frac{\tau}{2}\int_M hh' e^{-u} dV$ (although I don't know that u is uniquely defined, but I believe this is correct in a weak sense). From this, one obtains that $\mu(g_0,\tau) \leq \mu(g_t, e^{-t/\tau}\tau)$, for solutions $g_t = -2Ric$ to the Ricci flow [*N.B.* this formula isn't correct, see blog 8/20/03 for a correction]. One may then use this to prove a version of theorem 4.1 of [2] which gives a bound on the collapsing of g in terms of curvature and $\mu(g)$. Hopefully, I'll get to this next time.

References

[1] G. Perelman. Ricci flow with surgery on three-manifolds. math.DG/0303109.
[2] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. math.DG/0211159.