RESEARCH BLOG 7/15/03

I've been trying to understand Perelman's work, and so far I'm about through the first four sections of his paper [4], with a lot of help from Kleiner and Lott's notes.

In his second paper [3], Perelman claims that in his first paper [4], he verified that Ricci flow is a gradient flow for the first eigenvalue of $-4\Delta + R$. Indeed, the first section of the paper is called "Ricci flow as a gradient flow." But the difficulty with this is that there is an arbitrary function f which evolves in equation 1.1 as a backwards heat equation (so the evolution is not exactly a gradient), and some trickery is involved because of this to make things work (also, it is not exactly the Ricci flow, but a flow modified by a diffeomorphism which is a "gradient flow" in his context). There is some remarkable magic which makes the functional monotonic with respect to Ricci flow, but it seems that this functional was first discovered by string theorists. On p. 911 of the book "Quantum fields and strings: A course for mathematicians, vol. 2" [1], equations 6.69 are essentially the stationary version of what Perelman has in his paper as equations 1.1, with an extra field B which Perelman has eliminated, but kept the "dilaton field". His functional essentially appears in equation 6.74. It seems that Perelman discovered this functional from the string literature (maybe he attended this course at IAS?). The only other way I can guess he might have discovered it is by considering the gradient variation formulae for a metric with respect to the total scalar curvature, and with respect to the first eigenvalue of the Laplacian, and noticing that certain terms cancel when combined correctly. In the notes from Perelman's lectures at Stonybrook by Christine Sormani, he doesn't seem to indicate how he discovered this formula.

I want to interpret what Perelman might mean by Ricci flow being the gradient flow for the first eigenvalue of $-4\Delta + R$. First, what do we mean by gradient flow? Given a smooth manifold, the space of smooth Riemannian metrics is denoted $Riem(M)$, and is an open cone in the space of smooth symmetric bilinear forms S^2M on M, so that for $g \in Riem(M)$, $T_g(Riem(M)) \cong S^2M$. There is a natural inner product on $T_g(Riem(M))$, given by $(h,h')_g = \int_M h_{ik}h'_{lm}g^{il}g^{km}dv_g$ for $h,h' \in S^2M$ where we use the Einstein summation convention (h_{ik} is the tensor written in local coordinates, and g^{il} is the inverse tensor so that $g^{il}g_{lj} = \delta^i_j$). (Note: I think the convention is to usually assume that $g_{ij} = \delta_{ij}$ is diagonalized, in which case we can write $(h,h')_g = \int_M h_{ik}h'_{ik}dv_g$, summing over repeated indices). Thus, a flow $g_t = V(g)$, $V : Riem(M) \to S^2M$, is a gradient flow with respect to this L^2 inner product if there is a functional $\mathcal{F} : Riem(M) \to \mathbb{R}$ such that $\frac{\partial \mathcal{F}(g)}{\partial h} = (h, V(g))_g$, in which case $\frac{\partial \mathcal{F}(g_t)}{\partial t} = (V(g), V(g))_g \geq 0$, since $(,)_g$ is positive definite. This is of interest, since then the functional $\mathcal{F}$ is monotonic along the flow, and one might be able to use it to analyze the evolution of g_t under the flow. It turns out Perelman has shown that flows related to the Ricci flow are gradient flows, but with respect to modified L^2 metrics on $Riem(M)$.

We'll use R to denote the scalar curvature (suppressing the Riemannian structure (M,g)). The operator $-4\Delta + R$ is a Schrodinger operator (presumably because of the similarity with Schrodinger's equation, so R acts like a potential, but I would be interested if anyone could point me to the exact definition of a Schrodinger operator). It turns out to have a unique normalized positive eigenvector, that is there is a unique function $\Phi : M \to \mathbb{R}$ such that $\Phi(x) > 0$ for all $x \in M$, $-4\Delta\Phi + R\Phi = \lambda\Phi$, and $\|\Phi\|_2^2 = \int_M \Phi^2 dv_g = 1$ (if anyone knows a reference for this fact, please let me know - this is claimed on page 5 of Kleiner and Lott's notes). In fact, Φ has minimal eigenvalue for $-4\Delta + R$. For example, if R is constant (as for a Yamabe metric), then Φ is harmonic, so it must be the constant positive function such that $\|\Phi\|_2 = 1$. They also claim that the eigenvalue (and presumably the eigenfunction) depend smoothly on g. We will denote the eigenvalue λ and the eigenfunction $\Phi = e^{-f/2}$, following Perelman. Now, define an L^2 metric on $T_g(Riem(M))$ by $[h,h']_g = \frac{1}{2}\int_M h_{ik}h'_{lm}g^{il}g^{km}e^{-f}dv_g$. I claim that the flow $g_t = -2(Ric + \nabla^2 f)$, or written locally as $(g_{ij})_t = -2(R_{ij} + \nabla_i\nabla_j f)$, is the gradient flow for the functional $\mathcal{F}(g) = \int_M (R + |\nabla f|^2)e^{-f}dv_g = \int_M \Phi(-4\Delta\Phi + R\Phi)dv_g = \lambda$ with respect to the metric $[,]$.

To see this, note that the eigenvalue equation $-4\Delta\Phi + R\Phi = \lambda\Phi$ is equivalent to the equation $2\Delta f - |\nabla f|^2 + R = \lambda$ (this is claimed in 2.4 of [4], and follows easily from the computations in section 2 of [2]). The Rayleigh Ritz quotient method for obtaining the lowest eigenvalue of an operator $L = -4\Delta + R$ is given by

$$\lambda(g) = \inf_{\Phi} \frac{(\Phi, L\Phi)_2}{\|\Phi\|_2^2},$$

or normalizing $\lambda = \inf\{(\Phi, L\Phi)_2 | \|\Phi\|_2 = 1\}$ (where $(,)_2$ is the inner product on $L^2(M)$), in which case the eigenvalue equation can be seen as a generalization of Lagrange multipliers. On p. 5 of [4], we see the formula for the variation of $\mathcal{F}(g)$ with $\delta g_{ij} = v_{ij}$ and $\delta f = h$ as

$$\delta\mathcal{F}(v_{ij}, h) = \int_M e^{-f}[-v_{ij}(R_{ij}+\nabla_i\nabla_j f)+(v/2-h)(2\Delta f-|\nabla f|^2+R)]dv_g,$$

so that if we let f satisfy $\|\Phi\|_2^2 = \int_M e^{-f} dv_{g_t} = 1$ and $2\Delta f - |\nabla f|^2 + R = \lambda(g_t)$, then $\delta\|\Phi\|_2^2 = \int_M (v/2 - h)dv_g = 0$ (from 2.4 of [2]), and we have $\delta\mathcal{F} = [v_{ij}, -2(R_{ij} + \nabla_i\nabla_j f)]_g + \lambda \int_M (v/2 - h)e^{-f} dv_g = [v_{ij}, -2(R_{ij} + \nabla_i\nabla_j f)]_g$. Now, the flow $g_t = -2(Ric + \nabla^2 f)$ is equivalent to Ricci flow $g_t = -2Ric$ up to a diffeomorphism, since the term $\nabla^2 f$ only modifies the metric by the Lie derivative of the vector field ∇f. Notice that if one had smooth families of other eigenvectors (i.e. with non-minimal eigenvalues), then one would get gradient flows associated with inner products weighted by these eigenvectors. One may prove a formula analogous to the one given in Prop. 1.2 of [4] for the functional $\mathcal{F}(g) = \lambda$, that if $g_t = -2Ric$ exists for time T, then $\lambda(g_0) \leq \frac{n}{2T} Vol(M, g_0)$, so that in particular, if the Ricci flow existed for all time, then $\lambda(g_t) \leq 0$.

The results of section 3 of [4] have a similar interpretation in terms of eigenvalues and eigenvectors (as indicated in notes from his Stonybrook lectures). This result is quite remarkable, since Perelman is able to find a scale invariant quantity which is monotonic along the Ricci flow, which he then applies in section 4 to obtain a non-collapsing argument for solutions to Ricci flow. Consider the non-linear operator $L\Phi = -4\Delta\Phi + R\Phi - 2\Phi \ln \Phi$. This type of operator is proven in the paper [5] to have a well defined minimal "eigenvalue", that is a solution $L\Phi = \mu\Phi$, where $\mu \in \mathbb{R}$ is minimal. This is given by $\mu(g) = \inf\{(\Phi, L\Phi)_2 | \|\Phi\|_2 =$

$1\}$. In section 1 of [5], Rothaus proves that $\mu(g)$ exists, and that there exists a positive minimizer normalized Φ, satisfying $L\Phi = \mu\Phi$ (again, this seems to be a generalization of Lagrange multipliers). I conjecture that Φ is unique, and it depends smoothly on g.

Letting $\Phi = e^{-f/2}$, we get $\mu = W(g) = (\Phi, L\Phi)_2 = \int_M [|\nabla f|^2 + R - f]e^{-f} dv_g$, where $\|\Phi\|_2^2 = \int_M e^{-f} dv_g = 1$. If we consider $\delta W(g)[v_{ij}] = \int_M v_{ij}[-2(R_i j + \nabla_i \nabla_j f) + g_{ij}/2]e^{-f} dv_g$, so we get that $g_t = -2(Ric + \nabla^2 f) + g/2$, which is Ricci flow, modified by diffeomorphisms and scaling. Then Perelman defines $\nu(g) = \inf\{W(\tau g) | \tau > 0\}$, which is clearly a scale invariant quantity. He shows that if $\lambda > 0$, then ν is monotonic with respect to Ricci flow, which proves that there are no shrinking breathers with respect to Ricci flow. Next installment, I hope to expand on the argument in section 3.1 and the proof of Theorem 4.1 of [4].

References

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