Failure of Thompson Factorization and (some of) its descendants

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Stephen D. Smith
U. Illinois–Chicago

Thompson 80 Conference
Cambridge, (postponed from) September 2012

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Overview of the talk:

INTRODUCTION: The Frattini factorization.
1: The Thompson Factorization via the J-subgroup.
2: Determining when the factorization can fail (FF).
3: Pushing-up techniques (FF-modules in blocks).
4: Further factorizations: weak closure methods.
5: Oliver's conjecture on the J-subgroup (odd p).

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These slides can be found at:

www.math.uic.edu/∼smiths/talk.pdf

A note on references

This talk largely follows some exposition given by Aschbacher–Lyons–Smith–Solomon in [**?**]:

The Classification of Finite Simple Groups

(Surveys of the AMS, Vol. 172); especially from Sections B.6–B.8 there.

It also draws from expository material given in Aschbacher-Smith [**?**][**?**]:

The Classification of Quasithin Groups

(Surveys of the AMS, Vols. 111 and 112); especially from Chapters B, C, and E in [**?**].

Introduction: The Frattini Factorization

(An elementary situation yielding a factorization:)
If $G \geq T$ transitive on a G-orbit (say of α), then:

$$G = T \cdot G_\alpha$$

(Special case: The Frattini Argument (ca. 1885?):)
If $N \trianglelefteq G$ with $P \in \mathrm{Syl}_p(N)$, then:

$$G = N \cdot N_G(P)$$

(Subcase:) If V elem.ab. p-group $\trianglelefteq G$, then:

$$G = C_G(V) \cdot N_G(P)$$

Indeed for W weakly closed in P, $N_G(P) \leq N_G(W)$;
...and for $Z \leq V$, $C_G(V) \leq C_G(Z)$; so:

$$G = C_G(Z) \cdot N_G(W) \qquad \text{(FA)}$$

This form of Frattini arises in analysis of p-locals.
For example, often we will have $F^*(G) = O_p(G)$;
and then can take $V := \langle Z^G \rangle$ (get "p-reduced"),
where $Z := \Omega_1(Z(T))$, for $T \in \mathrm{Syl}_p(G)$.

§1: Thompson Factorization via $J(T)$

Thompson (1964) introduced:

$$J(T) := \langle A \text{ maxl-rank elem } \leq T \rangle.$$

Notice $J(T)$ is weakly closed in T
(and indeed in any R with $J(T) \leq R \leq T$).
So Frattini (FA) gives Thompson Factorization:

$$\text{If } J(T) \leq C_G(V),\ G = C_G(V) \cdot N_G\big(J(T)\big).$$

When MUST this good "If"-situation hold? E.g.:
Thompson (1966): for p-solvable G—unless
$p = 2$ or 3, with $SL_2(p)$ involved in G.
More generally, for the situation $F^*(G) = O_p(G)$,
where as mentioned earlier we can take $V := \langle Z^G \rangle$,
the desired factorization takes the form:

$$G = C_G\big(\ \Omega_1\big(Z(T)\big)\ \big) \cdot N_G\big(J(T)\big) \qquad \text{(TF)}$$

Note: Thompson triple-factorization methods (∼1972)
show roughly that "enough" local factorizations

§2: Failure of Thompson Factorization

If (TF) fails, some maxl-rank elem $A \not\leq C_G(V)$.
Since $|A| \geq |VC_A(V)|$, and $A \cap V \leq C_V(A)$, get:

$$\frac{|A|}{|C_A(V)|} \geq \frac{|V|}{|C_V(A)|}$$

I.e., $\overline{A} := A/C_A(V)$ is an "FF-offender".
There are various familiar cases of such $\overline{A}$, e.g.:
(a) (transvection) In V of dimension n,

$\overline{A}$ of rank 1 centralizing an $(n-1)$-subspace;
(b) any maximal unipotent radical of $GL(V)$:

$$\overline{U}_k := \left(\begin{array}{c|c} I_k & 0 \\ \hline * & I_{n-k} \end{array} \right)$$

The action of $\overline{U}_k$ is even *quadratic*. Indeed:
The Thompson Replacement Theorem (1969) shows that *any* FF-offender contains a quadratic offender.

(c) To see FF exhibited in a *local* subgroup G:

p-**solvable FF?** Glauberman (1973) showed the Thompson exceptions above are the only ones: Then $p = 2$ or 3, with G a product of terms $V_i L_i$, with V_i the natural module for $L_i \cong SL_2(p)$.

More general FF? Say $F^*(G) = O_p(G)$: Reduce to components $\overline{L}$ of $\overline{G} := G/C_G(V)$; i.e., take $\overline{G}$ to be quasisimple $\overline{L}$. Cooperstein-Mason (1978) gave the pairs $(V, \overline{L})$, but without proofs. Guralnick-Malle (2002) gave a more general treatment; in particular; $\overline{L}/Z(\overline{L})$ is either of Lie type in char p, or alternating with $p \leq 3$; and V is "small".

The list of FF-groups and modules is applied often in the Classification of Finite Simple Groups (CFSG).

To follow one important direction:

§3: Pushing-up (FF-modules in blocks)

Take $R \leq T$ with $R = O_p\big(N_G(R)\big)$. (Ex: $R = T$)
For any C char R, of course $N_G(R) \leq N_G(C)$.
Best, if we "push up" to $N_G(C)$ which is LARGER.
Failure? Set $C(G,R) := \langle N_G(C) : 1 < C \text{ char } R \rangle$,

$$C(G,R) \leq M < G, \qquad \text{(CPU)}$$

where we also assume R is Sylow in $\langle R^M \rangle$.

The Sylow$_2$ case: Take $p = 2, R = T$.
We might expect G narrow (e.g. small Lie rank?)
Also FF, if G local? E.g. [**?**, C.1.26];
roughly: if (TF) succeeds, then factors in $C(G,T)$.
Indeed Aschbacher's Local $C(G,T)$-Theorem (1981):
If $F^*(G) = O_2(G)$ and $C(G,T) < G$,
then $G = C(G,T)L_1 \cdots L_t$ for χ-blocks L_i.
Such a block has $L_2(2^m)$ or A_m (m odd), on V
with a UNIQUE nontrivial section (natural ... FF).
This led to Global $C(G,T)$-Theorem ($\sim$ 1982):
If G simple of char. 2 type with $C(G,T) < G$

And (CPU) with $R < T$? Get larger blocks...
Ex 1: Meierfrankenfeld-Stellmacher (1993):
R is rank-1 unipotent radical of rank-2 group...

Ex 2: The non-QT F_{23} "shadow" in QT [**?**]:
This has QT local $L = 2^{11} \cdot M_{23}$; START to elim...
Note there is $x \in 2^{11}$ with $C_L(x) = 2^{11} \cdot M_{22}$;
Indeed $C_{F_{23}}(x) \cong \langle x \rangle F_{22}$ (not QT).

The QT hyp's of [**?**] allow $L \trianglelefteq M$ maxl,
with $O_2(L) \geq V \cong 2^{11}$, and $\overline{L} \cong M_{23}$.
Further $R := O_2(LT)$ has $C(G, R) \leq M < G$.
Using M_{22}, this is inherited by $C_M(x) < C_G(x)$.
But under QT, no (CPU)-obstruction (like F_{22}).
We CONCLUDE $C_G(x) \leq M$ (at [**?**, 8.1.1]).
So NOT like in F_{23} (but L ruled out—yet).

§4: Weak-closure factorizations

EXPECT G-conj's of V in T in some max-rk A;
so that weak closure of V in T falls into $J(T)$.
Aschbacher (1981) variant of (TF) is based on w.cl.:

$$W_i := \langle A : A \leq T \cap V^g, m(V^g/A) = i \rangle;$$

$$C_i := C_T(W_i).$$

Values of "parameters" can give versions of (FA);
to ROUGHLY state 6.11.2 (cf. [**?**, B.8.5]):
Set $k := n(G)$ (involves groups over $\leq \mathbb{F}_{2^k}$);
assume $W_i > 1$ for i with $0 \leq i \leq s - a$. Then:

$$G = C_G(C_{i+k}) N_G(W_i). \tag{WC}$$

Uniqueness Case: Aschbacher (1983), to elim
almost strongly p-embedded maxl 2-local $M \geq T$,
from Thompson strategy get H with $T \leq H \not\leq M$.
With H, U as "G, V", (WC) gives $H = H_1 H_2$;
use uniqueness props of M, e.g. methods like (CPU),
to TRY to force $H \leq M$: contradict $H \not\leq M$.

References I