Math 310: **Final Exam** (Solutions)

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Problem 1: (a) Find the eigenvalues of $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

We see $\det(A - xI) = (3-x)[(1-x)^2 - 2\cdot 2] = (3-x)(x^2-2x-3) = (3-x)(x-3)(x+1)$.
So eigenvalues $-1, 3, 3$.

(b) Find the eigenspaces for each eigenvalue. Is A diagonalizable? (Why/why not?)
For -1 *get* $(-1,1,0)^T$*; for* 3*, span of* $(1,1,0)^T$ *and* $(0,0,1)^T$.
So YES diagonalizable (in particular, 2 lin.indep. eigenvectors for 3).

(c) Note A is symmetric—is it positive definite? Give *two* methods of deciding.
No. From above, not all eigenvalues are positive.
Alternatively, the three "upper left" determinants are $1, -3, -9$*, not all positive.*

Problem 2: (a) Give the general solution of the differential system $\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= -2y_1 + 4y_2 \end{aligned}$
.
Find eigenvalues 2 *and* 3*, and corresponding eigenvectors* $(1,1)^T$ *and* $(1,2)^T$.
$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$ *, so* $y_1 = c_1e^{2t} + c_2e^{3t}$ *and* $y_2 = c_1e^{2t} + 2c_2e^{3t}$.

(b) Give the particular solution when $y_1(0) = 3$ and $y_2(0) = 1$.
Putting in $t = 0$ *gives system with augmented matrix* $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 2 & 1 \end{array}\right)$,
Then Chapter 1 methods give solution $c_1 = 5$, $c_2 = -2$*;*
so $y_1 = 5e^{2t} - 2e^{3t}$, $y_2 = 5e^{2t} - 4e^{3t}$.

Problem 3: Let M be the Markov matrix $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

(a) Find the "steady state" eigenvector for M. (Components of vector should add to 1).
For eigenvalue 1, $M - 1I = \frac{1}{3}\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$,
so eigenvector is span of $(1,1)^T$. *Hence* $(\frac{1}{2}, \frac{1}{2})$ *is steady-state vector.*

(b) Diagonalize M and use this to get an expression for the power M^n.
Compute $\det(M - xI) = (x - \frac{1}{3})^2 - \frac{2}{3}^2 = x^2 - \frac{2}{3}x - \frac{1}{3} = (x-1)(x+\frac{1}{3})$.
So other eigenvalue is $-\frac{1}{3}$*; from* $M + \frac{1}{3}I = \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ *get eigenvector* $(1,-1)^T$.

Using $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$*, get* $S^{-1}MS = \Lambda$ *so* $M = S\Lambda S^{-1}$ *hence* $M^n = S(\Lambda^n)S^{-1}$.

Thus $M^n = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (-\frac{1}{3})^n \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$= \begin{pmatrix} 1 & (-\frac{1}{3})^n \\ 1 & -(-\frac{1}{3})^n \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + (-\frac{1}{3})^n & 1 - (-\frac{1}{3})^n \\ 1 - (-\frac{1}{3})^n & 1 + (-\frac{1}{3})^n \end{pmatrix}$.

Problem 4: Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. I *give* you that the eigenvalues are $3, 0, 0$.

(a) Find eigenvectors for these eigenvalues.

$A - 3I$ row-reduces to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, *so a 3-eigenvector is* $(1,1,1)^T$.

$A - 0I = A$ row-reduces to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

so independent 0-eigenvectors are $(1,-1,0)^T$ *and* $(1,0,-1)^T$.

(b) Note A is symmetric. So find *orthogonal* S with $S^{-1}AS$ diagonal.

(Remember this means the columns of S must be *orthonormal*).

We need to apply Gram-Schmidt to each eigenspace.

For 3, just divide by length to get $\frac{1}{\sqrt{3}}(1,1,1)^T$.

For 0, first is $(1,-1,0)^T$ *and second is (column form of)*

$$(1,0,-1) - (1,-1,0)\frac{1}{(1,-1,0)(1,-1,0)^T}(1,-1,0)(1,0,-1)^T$$

$$= (1,0,-1) - \tfrac{1}{2}(1,-1,0) = \tfrac{1}{2}(1,1,-2).$$

Divide by lengths to get $\frac{1}{\sqrt{2}}(1,-1,0)^T$ *and* $\frac{1}{\sqrt{6}}(1,1,-2)^T$.

So use $S = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$.