Math 310 (10am MWF): **Final Exam** (Solutions)
Prof. S. Smith: Wed 5 May 1999

You must SHOW WORK to receive credit.

Problem 1:
(a) (flashback to Hour Exam 1) In physics one sometimes sees the set S of 3×3 *skew*-symmetric matrices—satisfying $A^T = -A$. Notice such matrices have the general form $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$.

Show that S forms a subspace of the space $\mathbf{R}^{3\times 3}$ of all 3×3 matrices.

(closure,+) $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} + \begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix} = \begin{pmatrix} 0 & a+d & b+e \\ -a-d & 0 & c+f \\ -b-e & -c-f & 0 \end{pmatrix}$.

The sum is also skew, and so still lies in the set S.

(closure,sc.mult.) $f \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \begin{pmatrix} 0 & fa & fb \\ -fa & 0 & fc \\ -fb & -fc & 0 \end{pmatrix}$, *also skew, so in S.*

(b) (and to Exam 2:) Let S be the subspace of $\mathbf{R}^3$ spanned by $(1,1,1)^T$ and $(1,2,3)^T$. Find $S^\perp$.

Put in as rows of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, *solve $Ax = 0$ to get $x_3(1,-2,1)^T$.*

Problem 2: Let $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. (a) Find the eigenvalues of A.

$\det(A - xI) = x^2 - 4 = (x-2)(x+2)$ *so eigenvalues are* $2, -2$.

(b) Find the eigenspaces for those eigenvalues. Is A diagonalizable? Why/why not?

For 2: Solve $(A - 2I)x = 0$ with $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ *to get $a(1,1)^T$;*

For -2: $(A + 2I)x = 0$ with $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ *to get $b(1,-1)^T$.*

Diagonalizable—distinct eigenvalues (not repeated).

Problem 3: Given the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & y_1 & + & 3y_2 \\ y_2' & = & 3y_1 & + & y_2 \end{pmatrix}$.

GIVEN: the matrix $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ has eigenvalues $4, -2$; with eigenvectors $(1,1)^T$ and $(1,-1)^T$.

(a) Give the *general* solution of the system (with undetermined constants c_1, c_2).

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^{4t} \\ c_2 e^{-2t} \end{pmatrix}$ *so $y_1 = c_1 e^{4t} + c_2 e^{-2t}$ and $y_2 = c_1 e^{4t} - c_2 e^{-2t}$.*

(b) Now determine the values of c_1, c_2 for the initial value problem $y_1(0) = 8$, $y_2(0) = 4$.

Solve $\left(\begin{array}{cc|c} 1 & 1 & 8 \\ 1 & -1 & 4 \end{array}\right)$ *to get $c_1 = 6, c_2 = 2$. So $y_1 = 6e^{4t} + 2e^{-2t}$ and $y_2 = 6e^{4t} - 2e^{-2t}$.*

Problem 4: (a) Let A be the (Markov) matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}$.
Indicate the "steady-state" vector; that is, a vector v such that $Av = v$—and the coordinates of v add up to 1.

We need an eigenvector for eigenvalue 1. Solve $(A - 1.I)x = 0$ *using* $\begin{pmatrix} -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{2}{3} \end{pmatrix}$, *to get solutions* $a(2,3)^T$; *so use vector* $\frac{1}{5}(2,3)^T$.

(b) Is the symmetric matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ positive definite? Why/why not?

No: for example, determinants of principal minors are 1,-1,0; not all positive.

Problem 5: (a) Let A be the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. GIVEN: A has eigenvalues $0, 1, 2$.
Find eigenvectors, and diagonalize A (that is, give X such that $X^{-1}AX$ is diagonal).

For 0: *get* $a(1,0,-1)^T$. *For* 1: *get* $b(0,1,0)^T$ *For* 2: *get* $c(1,0,1)^T$. *So* $X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

(b) The matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues $2, 0$ and eigenvectors $(1,1)^T$ and $(1,-1)^T$. Find e^A.

We can diagonalize A *with* $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. *So from* $A = XDX^{-1}$ *we have* $e^A = Xe^DX^{-1} =$
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^0 = 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^2+1 & e^2-1 \\ e^2-1 & e^2+1 \end{pmatrix}$$

Problem 6: Let A be the symmetric matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. GIVEN: A has eigenvalues $1, 1, 4$.

(a) Find a basis for each eigenspace.

For 4: find eigenspace spanned by $(1,1,1)^T$. *For 4: Get eigenvectors* $(-b-c, b, c)^T$.
So one basis is $(1,-1,0)^T$ *and* $(1,0,-1)^T$.

(b) Use Gram-Schmidt to find an *orthonormal* basis for each eigenspace. Use that to build an orthogonal matrix X (that is, $X^{-1} = X^T$) with $X^{-1}AX$ diagonal.

For 4, *just use* $\frac{1}{\sqrt{3}}(1,1,1)^T$. *For* 1, *apply Gram-Schmidt to get* $\frac{1}{\sqrt{2}}(1,-1,0)^T$ *and* $\frac{1}{\sqrt{6}}(1,1,-2)^T$.

So can use $X = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$.