Math 310: **Hour Exam 1** (Solutions)
Prof. S. Smith: Fri 18 October 2002

Problem 1: (a) Using Gauss-Jordan elimination, find the row-reduced echelon form of the following augmented matrix: $(A|b) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right)$. Show the STEPS of the method you use.

Use row operations (Gauss-Jordan)

$$\overset{A_2^{-1\times1}\ A_3^{-4\times1}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \overset{M_{-1\times2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right) \overset{A_1^{-2\times2}\ A_3^{-1\times2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b) Using your answer in (a), give all SOLUTIONS of the linear equation system $Ax = b$ determined by the augmented matrix $(A|b)$.

Notice column 3 has no pivot, so that variable is free.
Infinite number of solutions: $(2+\alpha, -1-\alpha, \alpha)^T$ *for real* α.

Problem 2: (a) Find the LU-decomposition (show method) of the matrix $A = \begin{pmatrix} 3 & 4 \\ 6 & 7 \end{pmatrix}$.

To row-reduce (Gaussian elmination) $\begin{pmatrix} 3 & 4 \\ 6 & 7 \end{pmatrix}$, *we apply* $A_2^{-2\times1}$ *to obtain* $\begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix}$ *as* U, *and take as* L *the inverse of the matrix for* $A_2^{-2\times1}$, *namely* $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

(b) Find the inverse (by any method—but show work) of the matrix $A = \begin{pmatrix} 3 & 4 \\ 6 & 7 \end{pmatrix}$.

By adjoint method, $A^{-1} = -\frac{1}{3}\begin{pmatrix} 7 & -4 \\ -6 & 3 \end{pmatrix}$.

Problem 3: (a) Recall that $\mathcal{P}_3$ is the space of polynomials of degree less than three (that is, quadratic polynomials). Are the three "vectors" $1+x+x^2$, $2-2x+2x^2$, and $4+4x^2$ in this space linearly independent ? (Why/why not ?)

Set a linear combination equal to zero: $a(1+x+x^2) + b(2-2x+2x^2) + c(4+4x^2) = 0$.
Get an equation for each power of x:
(1:) $a + 2b + 4c = 0$
(x*:)* $a - 2b = 0$
(x^2*:)* $a + 2b + 4c = 0$

When we row-reduce the corresponding matrix $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & -2 & 0 \\ 1 & 2 & 4 \end{pmatrix}$ *to* $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, *the third column has no pivot: so "No", linearly dependent rather than independent.*

(b) Solve $Ax = b$ by Cramer's Rule for the augmented matrix $(A|b)$: $\left(\begin{array}{cc|c} 2 & 4 & 6 \\ 3 & 5 & 7 \end{array}\right)$.

$\det(A) = 2\cdot5 - 3\cdot4 = -2$, *so* $x_1 = \det\begin{pmatrix} 6 & 4 \\ 7 & 5 \end{pmatrix}/(-2) = -1$, *and* $x_2 = \det\begin{pmatrix} 2 & 6 \\ 3 & 7 \end{pmatrix}/(-2) = 2$.

Problem 4: (a) Let S be the subspace of $\mathbf{R}^3$ consisting of all 3-vectors whose coordinates sum to zero: that is, $x_1 + x_2 + x_3 = 0$. Show that S is a subpace of $\mathbf{R}^3$.

(+) If (x_1, x_2, x_3) *satisfies* $x_1 + x_2 + x_3 = 0$, *and* (y_1, y_2, y_3) *satisfies* $y_1 + y_2 + y_3 = 0$, *then their sum* $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$ *satisfies*

$$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0 + 0 = 0,$$

and so also lies in S.

(sc.mult.) For a scalar c, *the scalar multiple* $c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3)$ *satsifies* $cx_1 + cx_2 + cx_2 = c(x_1 + x_2 + x_3) = c \cdot 0 = 0$, *and so also lies in* S.

(b) Give a basis for the row space of the matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{pmatrix}$.

What is the dimension of this space ?

We compute the row-reduced echelon form of A *as* $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Therefore the first two rows give a basis, and the dimension is 2.

Problem 5: (a) Find a basis for the subspace S of polynomials in $\mathcal{P}_2$ of form $ax^2 + bx + (3a + 2b)$; show that your basis IS a basis.

Best to choose the "free variables" a, b *in the "standard basis" way:*

Thus $a = 1, b = 0$ *gives* $x^2 + 3$, $a = 0, b = 1$ *gives* $x + 2$.

Spanning set: $ax^2 + bx + (3a + 2b) = a(x^2 + 3) + b(x + 2)$.

linearly independent: $0 = ax^2 + bx + (3a + 2b)$ *gives equations for each power of* x, *including* $a = 0$, $b = 0$; *that is, only the zero-solution.*

(b) Find the matrix of transition from the "old" basis given by the standard basis of $\mathbf{R}^2$ (namely $(1, 0)^T$ and $(0, 1)^T$) to the "new" basis given by $(1, 2)^T$ and $(2, 3)^T$.

One way: The matrix $[new]_{old}$ *is given by* $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$,

so the transition matrix $[old]_{new}$ *from old to new is given by its inverse, namely* $\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$.

What are the coordinates of $(5, 7)^T$ in this new basis?

Can multiply transition matrix by "old" coordinates $(5, 7)^T$ *to get new coordinates* $(-1, 3)^T$.