

A DIMENSION-DEPENDENT MAXIMAL INEQUALITY

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ABSTRACT. In this short note we show that $\sup\{\|M_\nu\| : \nu \text{ is measure on } \mathbb{R}^n\}$, where $\|M_\nu\|$ denotes the centered Hardy-Littlewood maximal operator, depends exponentially on n .

1. STATEMENT OF THE PROBLEM

Let ν be a σ -finite measure on the Borel subsets of \mathbb{R}^n . Define the Hardy-Littlewood centered maximal operator associated with ν by

$$M_\nu f(x) = \sup_{r>0} \frac{1}{\nu(B_r(x))} \int_{B_r(x)} |f| d\nu, \quad x \in \mathbb{R}^n.$$

It was proved in [1, 2] that

$$\|M_\nu f\|_{L_p(\mathbb{R}^n, \nu)} \leq C \|f\|_{L_p(\mathbb{R}^n, \nu)}, \quad 1 < p < \infty,$$

where C does not depend on ν . We present a simple construction showing that C depends exponentially on n . This answers the question posed in [2, 3].

2. CONSTRUCTION

Claim 2.1. *There is an absolute constant $\alpha > 1$ such that one can find $[\alpha^n]$ points $x_1, x_2, \dots, x_{[\alpha^n]}$ on the euclidian sphere S^{n-1} such that*

$$\|x_i - x_j\| > 1, \quad i \neq j.$$

The maximal value of α is immaterial. A simple argument based on volume estimates yields $\alpha \geq e^{(\pi/6)^{21/2}}$.

Let us fix $x_1, x_2, \dots, x_{[\alpha^n]}$ as in the claim and put

$$\nu = \delta_{\{0\}} + \sum_i \delta_{\{x_i\}}.$$

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Define $f = \delta_{\{0\}}$. Then $\|f\|_{L_p(\mathbb{R}^n, \nu)} = 1$. On the other hand,

$$(M_\nu f)(x_i) \geq \frac{1}{\nu(B_1(x_i))} \int_{B_1(x_i)} |f| d\nu = 1/2, \quad i = 1, \dots, [\alpha^n].$$

Hence, $\|M_\nu f\|_{L_p(\mathbb{R}^n, \nu)} \geq \frac{1}{2}[\alpha^n]$.

This is the end of construction.

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