Model Theory and Differential Galois Theory

David Marker

University of Illinois at Chicago

marker@math.uic.edu

http://www.math.uic.edu/~marker

Differentially Closed Fields

We work with differential fields $(k, +, \cdot, \delta)$. $C(k) = \{x \in k : \delta x = 0\}$

• We say that a differential field (K, δ) is *differentially closed* if every finite system of algebraic differential equations that has a solution in an extension field already has a solution in K.

• Every differential field k has a differential closure K (i.e. a differentially closed $K \supseteq k$ such that if $L \supseteq k$ is differentially closed there is a differential embedding of K into L fixing k.) The differential closure is unique up to isomorphism.

• (Quantifier Elimination in DCF)

$$\text{definable} = \text{Kolchin-constructible}$$

• (Universal Domain) We let $\mathbb{K}$ be a large universal differentially closed field. All fields we consider will be small subfields of $\mathbb{K}$.

Kolchin's Galois Theory

Definition 1 Let k and l be differential fields with $k \subseteq l$. We say that l/k is *strongly normal* if and only if:

i) $C(l) = C(k)$ is algebraically closed;

ii) l/k is finitely generated;

iii) if $\sigma : \mathbb{K} \to \mathbb{K}$ is a differential automorphism fixing k pointwise, then $\langle l, C(\mathbb{K})\rangle = \langle \sigma(l), C(\mathbb{K})\rangle$.

Examples i) Picard-Vessiot extensions,

ii) Weierstrass extensions: $l = k\langle y\rangle$, where y is a nonconstant solution to $(y')^2 = a^2(y^3 - y)$ and $C(k) = C(l)$.

Theorem 2 (Kolchin) *If l/k is strongly normal, then there is an algebraic group G defined over $C(k)$ such that* $\mathrm{Gal}_\delta(l/k) = G(C(k))$.

Poizat's Model Theoretic Proof

Suppose $l = k\langle \mathbf{a} \rangle$ is strongly normal over k where $\mathbf{a} = (a_1, \ldots, a_n)$. Let K be a differential closure of k. Note that $C(K) = C(k)$.

Step 1 Show that $l \subseteq K$.

Step 2 Find a definable $X \subseteq \mathbb{K}^n$ such that $\mathbf{b} \in X(K)$ if and only if there is a differential automorphism σ of $\mathbb{K}$ fixing k with $\sigma(\mathbf{a}) = \mathbf{b}$. This is a special case of a general model theoretic fact.

- (Isolation) If K is a differential closure of k and $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$, there is a k-definable set $X \subseteq \mathbb{K}^n$ such that $\mathbf{b} \in X$ iff there is a differential automorphism $\sigma : \mathbb{K} \to \mathbb{K}$ fixing k with $\sigma(\mathbf{a}) = \mathbf{b}$.

Note: If $\sigma : \mathbb{K} \to \mathbb{K}$ is a differential automorphism fixing k and $\sigma(\mathbf{a}) \in K$, then $\sigma | l \in \mathrm{Gal}_\delta(l/k)$.

<u>Step 3</u> Show there is a k-definable function g such that for all $\mathbf{b} \in X$, there is $\mathbf{c} \in C(\mathbb{K})^m$ such that $g(\mathbf{a}, \mathbf{c}) = \mathbf{b}$. [This is just the fact that $\mathbf{b} \in \langle l, C(\mathbb{K})\rangle$].

<u>Step 4</u> Let $R(\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$ hold if $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3} \in X(K)$ and $\sigma, \tau : \mathbb{K} \rightarrow \mathbb{K}$ are differential automorphisms fixing k pointwise with $\sigma(\mathbf{a}) = \mathbf{b_1}$ and $\tau(\mathbf{a}) = \mathbf{b_2}$, then $\sigma \circ \tau(\mathbf{a}) = \mathbf{b_3}$. $R(\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$ holds iff

$\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3} \in X$ and $\exists \mathbf{c} \in C(K)^m$ $g(\mathbf{a}, \mathbf{c}) = \mathbf{b_2}$ $g(\mathbf{b_1}, \mathbf{c}) = \mathbf{b_3}$.

- (X, R) is a definable group isomorphic to $\mathrm{Gal}_\delta(l/k)$.

We have already shown that $\mathrm{Gal}_\delta(l/k)$ is isomorphic to a group definable in K.

Step 5 Let $Y = \{\mathbf{c} \in C(K)^m : g(\mathbf{a}, \mathbf{c}) \in X(K)\}$. Let $\mathbf{c} \sim \mathbf{c_1}$ iff $g(\mathbf{a}, \mathbf{c}) = g(\mathbf{a}, \mathbf{c_1})$. Then $\mathrm{Gal}_\delta(l/k)$ is isomorphic to a group definable on $Y/\sim$.

Step 6 Any set definable in $C(K)^m$ is definable without using δ and hence constructible. Thus Y and $\sim$ are constructible.

We now use two important fact from the model theory of algebraically closed fields (or classical algebraic geometry).

- $Y/\sim$ is constructible.

- Any constructible group is constructibly isomorphic to an algebraic group.

Differential Algebraic Groups

Differential algebraic groups are groups where the underlying set is an (abstract) differential algebraic variety and multiplication and inverse are differential morphisms.

Example 3

$$\left\{ \begin{pmatrix} x & \delta x \\ 0 & x \end{pmatrix} : x \in K^{\times} \right\}$$

Differential algebraic groups are definable.

Theorem 4 (Pillay) *i) Every definable group in $\mathbb{K}$ is definably isomorphic to a differential algebraic group.*

ii) Every connected differential algebraic group can be embedded as a Kolchin closed subgroup of an algebraic group.

Pillay's Differential Galois Theory

Idea: Replace the constant field by an arbitrary definable set.

Definition 5 Let k be a differential field and $X \subseteq \mathbb{K}^n$ is a Kolchin closed set defined over k. We say that l/k is X-*strongly normal* if

i) $X(k) = X(K)$ for K a differential closure of l;

ii) l is finitely generated over k;

iii) If $\sigma : \mathbb{K} \to \mathbb{K}$ is a differential automorphism fixing k pointwise then $\langle l, X(\mathbb{K})\rangle = \langle \sigma(l), X(\mathbb{K})\rangle$.

strongly normal = C-strongly normal

• Suppose X is a Kolchin-closed set defined over k. We let $k\langle X\rangle$ be the field of differential rational functions on X. We say X is *finite dimensional* if the transcendence degree of $k\langle X\rangle$ over k is finite.

l/k is *generalized strongly normal* if X-strongly normal for some X.

Theorem 6 (Pillay) *i) If l/k is generalized strongly normal, then there is a finite dimensional differential algebraic group G defined over k such that* $\mathrm{Gal}_\delta(l/k)$ *is isomorphic to $G(k)$.*

ii) There is a Galois correspondence between intermediate differential subfields of l/k and differential algebraic subgroups of G defined over k.

Theorem 7 (Pillay) *Suppose k is a differential field and K is a differential closure of k. Then l/k is generalized strongly normal iff $l \subseteq K$ and there is a k-definable finite dimensional differential algebraic group G and a principle homogeneous space X for G such that $G(k) = G(K)$ and $l = k\langle \mathbf{a} \rangle$ for some $\mathbf{a} \in X$.*

Theorem 8 (Pillay) *Let G be a connected finite dimensional differ tial algebraic group. There are differential fields $k \subseteq l \subset \mathbb{K}$ such that G is defined over k, l/k is generalized strongly normal, and* $\mathrm{Gal}_\delta(l/k)$ *is isomorphic to $G(k)$.*

What about inverse problems?

Buium–Manin kernels

Theorem 9 *Suppose A is a simple Abelian variety of dimension d. There is a differential algebraic group homomorphism $\mu : A \to \mathbb{K}^d$ a differential algebraic group homomorphism such that $A^\# = \ker \mu$ is a finite dimensional differential algebraic group. In fact $A^\#$ is the Kolchin-closure of the torsion points of A.*

• If A is defined over C, then μ is Kolchin's logarithmic derivative and $A^\# = A(C)$.

• If A is not isomorphic to any Abelian variety defined over C, then $A^\#$ is **very** different from an algebraic group. For example, any Kolchin-constructible subset of $(A^\#)^n$ is a finite Boolean combination of cosets. In particular any Kolchin-constructible subset of $A^\#$ is finite or co-finite.

Example 10 *Let K be the differential closure of $\mathbb{C}(t)$ where $\delta t = 1$. Let E be the elliptic curve $Y^2 = X(X-1)(X-t)$. Then $E^{\#}(K) = Tor(E)$.*

Some Inverse Problems

Theorem 11 (M–Pillay) *Suppose k is a differential field such that k is the algebraic closure of a finitely generated extension of $C(k)$ and $C(k)$ has infinite transcendence degree. Let A be a simple abelian variety defined over k. Then k has a generalized strongly normal extension l/k with $\mathrm{Gal}_\delta(l/k) \cong A^\#(k)$.*

Lemma 12 *The Buium–Manin homomorphism $\mu : A(k) \to k^d$ is not surjective.*

Let $\mathbf{a} \in k^d \setminus \mu(A(k))$. Then $\mu^{-1}(\mathbf{a})$ is a principle homogeneous space for $A^\#$.

Suppose K be a differential closure of k and $\mathbf{b} \in A(K)$ with $\mu(\mathbf{b}) = \mathbf{a}$.

Let $l = k\langle \mathbf{b} \rangle$.