Math 494: Topics in Algebra
Problem Set 3

Due Wednesday March 26: Do 5 of the following problems.

1) Let $L \subseteq \mathbb{P}_2(k)$ be a projective line. Show that there are homogeneous polynomials $f, g, h \in k[X, Y]$ of degree one such that

$$[(x, y)] \mapsto [(f(x, y), g(x, y), h(x, y))]$$

is a well-defined bijection between $\mathbb{P}_1(k)$ and L.

2) Let $p \in \mathbb{P}_2(k)$. Let $\mathcal{L}_p$ be the set of all lines $L \subseteq \mathbb{P}_2(k)$ with $p \in L$. Find a natural bijection between $\mathbb{P}_1(k)$ and $\mathcal{L}_p$.

3) Let $C \subseteq \mathbb{P}_2(\mathbb{C})$ be the curve

$$Y^3 = X^3 + X^2 Z - Z^3$$

and let $L \subseteq \mathbb{P}_2(k)$ be the line $X - Y = 0$. Find all points of $C \cap L$.

4) a) Suppose $p_1, \ldots, p_N \in \mathbb{P}_2(k)$ where $N \leq \frac{d^2+3d}{2}$. Then there is a curve C of degree d with $p_1, \ldots, p_N \in C$.

b) Find 6 points $p_1, \ldots, p_6$ in $\mathbb{P}_2(\mathbb{C})$ such that there is no conic C with $p_1, \ldots, p_6 \in C$.

5) Suppose K is an algebraically closed field and $f \in K[X, Y]$ is a nonconstant polynomial. Prove that $V(f)$ is infinite.

6) Prove Theorem 4.19 from §4 of the notes.

Bonus Problem: Suppose $f(X, Y, Z) = X^2 + aY^2 + bZ^2$ where $a, b \in \mathbb{Q}$. Prove that either

i) $|C \cap \mathbb{P}_2(\mathbb{Q})| = \emptyset$ or

ii) $|C \cap \mathbb{P}_2(\mathbb{Q})|$ is infinite.

Show by example that both i) and ii) are possible.