4 The Hanf Number of $\mathcal{L}_{\omega_1,\omega}$

In Exercise 1.13 we showed that the Upward Löwenheim-Skolem Theorem fails for $\mathcal{L}_{\omega_1,\omega}$ by giving a sentence with models of size $2^{\aleph_0}$ but no larger models. In this section we will show that their is a cardinal κ such that for all $\phi \in \mathcal{L}_{\omega_1,\omega}$ if ϕ has a model of cardinality κ, then ϕ has models of all infinite cardinalities. We call such a κ the *Hanf number* of $\mathcal{L}_{\omega_1,\omega}$. It is general nonsense that there is a Hanf number.

Exercise 4.1 Let I be a set. For each $i \in I$, let K_i be a class of models. Let $\mathcal{K} = \{K_i : i \in I\}$. Prove that there is κ such that for all i if K_i has a model of size κ, then K_i has aribitrarily large models. The least such κ is the Hanf number for $\mathcal{K}$.

For κ an infinite cardinal and α an ordinal, we inductively define $\beth_\alpha(\kappa)$ by $\beth_0(\kappa) = \kappa$ and

$$\beth_\alpha(\kappa) = \sup_{\beta<\alpha} 2^{\beth_\beta(\kappa)}.$$

In particular $\beth_1(\kappa) = 2^\kappa$. We let $\beth_\alpha = \beth_\alpha(\aleph_0)$. Under the Generalized Continuum Hypothesis, $\beth_\alpha = \aleph_\alpha$.

Our main theorem, due to Morley, is that $\beth_{\omega_1}$ is the Hanf number for $\mathcal{L}_{\omega_1,\omega}$.

Theorem 4.2 *If for all $\alpha < \omega$ there is $\mathcal{M} \models \phi$ with $|M| \geq \beth_\alpha$, then ϕ has models of all infinite cardinalities.*

The next exercise generalizes Exercise 1.13 to show that this result is optimal.

Exercise 4.3 Let $\alpha < \omega_1$. Let $\mathcal{L} = \{U_\beta : \beta \leq \alpha + 1\} \cup \{E\} \cup \{c_0, c_1, \ldots, \}$, where U_β is a unary relation and E is binary. Let ϕ assert that:

i) $U_0 = \{c_0, c_1, \ldots\}$ and $\forall x\ x \in U_{\alpha+1}$.
ii) $U_\gamma \subseteq U_\beta$ for $\gamma < \beta$, and everything is in $U_{\alpha+1}$;
iii) $U_\beta = \bigcup_{\gamma<\beta} U_\gamma$ for $\beta < \alpha$ a limit ordinal;
iv) if $x \in U_{\beta+1} \setminus U_\beta$ and $E(y, x)$ then $y \in U_\beta$;
v) (extensionality) if $\{x : E(x, y)\} = \{x : E(x, z)\}$, then $y = z$.

a) Show that there is $\mathcal{M} \models \phi$ with $|M| = \beth_{\alpha+1}$. [Hint: Let $U^{\mathcal{M}}_{\beta+1} = U^{\mathcal{M}}_\beta \cup \mathcal{P}(U^{\mathcal{M}}_\beta)$ and $U^{\mathcal{M}}_\beta = \bigcup_{\gamma<\beta} U^{\mathcal{M}}_\gamma$ for γ a limit ordinal.]
b) Show that every model of ϕ has cardinality at most $\beth_{\alpha+1}$.

We can refine these questions by looking at complete sentences ϕ.

Exercise 4.4 Let $\mathcal{L} = \{+, 0, G_1, G_2, \ldots\}$. Let ϕ be a $\mathcal{L}_{\omega_1,\omega}$-sentence asserting that:

i) we have a group where every element has order 2;
ii) G_1 is an index 2 subgroup and each G_{n+1} is an index 2 subgroup of G_n;
iii) $\bigcup G_n = \{0\}$.

Prove that ϕ is complete and every model has size at most $2^{\aleph_0}$.

Baumgartner [4], building in work of Malitz [8], the Hanf number for complete $\mathcal{L}_{\omega_1,\omega}$-sentences is still $\beth_{\omega_1}$. Hjorth [7] has given an examples for $\alpha < \omega_1$ of complete sentences with models of size $\aleph_\alpha$ but no larger model.

Here is the main idea of the proof of Theorem 4.2.

• By expanding the language we may assume that we have $\phi \in T$ where T is a theory in a countable fragment with built in Skolem functions.

• Under the assumptions of Theorem 4.2 we can have find a model of T with an infinte set of indiscernibles.

• Taking Skolem hulls we get models of all infinite cardinalities.

The first and third steps are routine. Finding a model with indiscernibles needs a generalization of Ramsey's Theorem.

The Erdos-Rado Partition Theorem

For X a set and κ, λ (possibly finite) cardinals, we let $[X]^\kappa$ be the collection of all subsets of X of size κ. We call $f : [X]^\kappa \to \lambda$ a *partition* of $[X]^\kappa$. We say that $Y \subseteq X$ is *homogeneous* for the partition f if there is $\alpha < \lambda$ such that $f(A) = \alpha$ for all $A \in [Y]^\kappa$ (i.e., f is constant on $[Y]^\kappa$). Finally, for cardinals κ, η, μ, and λ, we write $\kappa \to (\eta)^\mu_\lambda$ if whenever $|X| \geq \kappa$ and $f : [X]^\mu \to \lambda$, then there is $Y \subseteq X$ such that $|Y| \geq \eta$ and Y is homogeneous for f.

In this notation Ramsey's Theorem can be stated as $\aleph_0 \to (\aleph_0)^n_k$.

When we begin partitioning sets into infinitely many pieces it becomes harder to find homogeneous sets.

Proposition 4.5 $2^{\aleph_0} \not\to (3)^2_{\aleph_0}$.

Proof We define $F : [2^\omega]^2 \to \omega$ by $F(\{f, g\})$ is the least n such that $f(n) \neq g(n)$. Clearly, we cannot find $\{f, g, h\}$ such that $f(n) \neq g(n)$, $g(n) \neq h(n)$, and $f(n) \neq h(n)$.

In fact $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$. This is a special case of a useful and powerful generalization of Ramsey's Theorem.

Theorem 4.6 (Erdös–Rado Theorem) $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$.

Proof We prove this by induction on n. For $n = 0$, $\kappa^+ \to (\kappa^+)^1_\kappa$ is just the Pigeonhole Principle.

Suppose that we have proved the theorem for $n - 1$. Let $\lambda = \beth_n(\kappa)^+$, and let $f : [\lambda]^{n+1} \to \kappa$. For $\alpha < \lambda$, let $f_\alpha : [\lambda \setminus \{\alpha\}]^n \to \kappa$ by $f_\alpha(A) = f(A \cup \{\alpha\})$.

We build $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_\alpha \subseteq \ldots$ for $\alpha < \beth_{n-1}(\kappa)^+$ such that $X_\alpha \subseteq \beth_n(\kappa)^+$ and each X_α has cardinality at most $\beth_n(\kappa)$. Let $X_0 = \beth_n(\kappa)$. If α is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

Suppose we have X_α with $|X_\alpha| = \beth_n(\kappa)$. Because

$$\beth_n(\kappa)^{\beth_{n-1}(\kappa)} = (2^{\beth_{n-1}(\kappa)})^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa),$$

there are $\beth_n(\kappa)$ subsets of X_α of cardinality $\beth_{n-1}(\kappa)$. Also note that if $Y \subset X_\alpha$ and $|Y| = \beth_{n-1}(\kappa)$, then there are $\beth_n(\kappa)$ functions $g : [Y]^n \to \kappa$ because

$$\kappa^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa).$$

Thus, we can find $X_{\alpha+1} \supseteq X_\alpha$ such that $|X_{\alpha+1}| = \beth_n(\kappa)$, and if $Y \subset X_\alpha$ with $|Y| = \beth_{n-1}(\kappa)$ and $\beta \in \lambda \setminus Y$, then there is $\gamma \in X_{\alpha+1} \setminus Y$ such that $f_\beta|[Y]^n = f_\gamma|[Y]^n$.

Let $X = \bigcup_{\alpha<\beth_{n-1}(\kappa)^+} X_\alpha$. If $Y \subset X$ with $|Y| \leq \beth_{n-1}(\kappa)$, then $Y \subset X_\alpha$ for some $\alpha < \beth_n(\kappa)^+$. If $\beta \in \lambda \setminus Y$, then there is $\gamma \in X \setminus Y$ such that $f_\beta|[Y]^n = f_\gamma|[Y]^n$.

Fix $\delta \in \lambda \setminus X$. Inductively construct $Y = \{y_\alpha : \alpha < \beth_{n-1}^+(\kappa)\} \subseteq X$. Let $y_0 \in X$. Suppose that we have constructed $Y_\alpha = \{y_\beta : \beta < \alpha\}$. Choose $y_\alpha \in X$ such that $f_{y_\alpha}|[Y_\alpha]^n = f_\delta|[Y_\alpha]^n$.

By the induction hypothesis, there is $Z \subseteq Y$ such that $|Z| \geq \kappa^+$ and Z is homogeneous for f_δ. Say $f_\delta(B) = \gamma$ for all $B \in [Z]^n$. We claim that Z is homogeneous for f. Let $A \in [Z]^{n+1}$. There are $\alpha_1 < \ldots < \alpha_{n+1}$ such that $A = \{y_{\alpha_1}, \ldots, y_{\alpha_{n+1}}\}$. Then

$$f(A) = f_{y_{\alpha_{n+1}}}(\{y_{\alpha_1}, \ldots, y_{\alpha_n}\}) = f_\delta(\{y_{\alpha_1}, \ldots, y_{\alpha_n}\}) = \gamma.$$

Thus, Z is homogeneous for f.

Corollary 4.7 $\beth_{\alpha+n}^+ \to (\beth_\alpha^+)^{n+1}_{\beth_\alpha}$.

Proof This follows from Erdös–Rado because $\beth_{\alpha+n} = \beth_n(\beth_\alpha)$.

Constructing Indiscernibles

We can now prove Theorem 4.2.

As in the proof of the Downward Löwenheim-Skolem Theorem in 1.10 we can assume we can expand our language $\mathcal{L}$ so that we may assume that:

- there is a countable fragment $\mathcal{F}$ of $\mathcal{L}_{\omega_1,\omega}$ and $T \subseteq \mathcal{F}$ a theory with built in Skolem Functions such that $\phi \in T$ and T has models of cardinality $\beth_\alpha$ for all $\alpha < \omega_1$;
- $\mathcal{L}$ contains two disjoint countably infinite sets of constant symbols C and $D = \{d_0, d_1, \ldots\}$.

Let $\Gamma = \{d_i \neq d_j : i \neq j\} \cup \{\theta(d_{i_1}, \ldots, d_{i_m}) \leftrightarrow \theta(d_{j_1}, \ldots, d_{j_m}) : \theta(v_1, \ldots, v_m) \in \mathcal{F}, i_1 < \ldots i_m, j_1 < \ldots, j_m\}$.

If we can find $\mathcal{M} \models T \cup \Gamma$, then the interpretation of D gives us a set of indiscernibles. By the usual techniques we can stretch the indiscernibles to build arbirtrarily large models of T.

Let Σ be the set of all finite sets σ of formulas from $\mathcal{F}$ using only finitely many free variables from $C \cup D$ such that $\sigma(\overline{c}, d_1, \ldots, d_n) \in \Sigma$ if and only if there are arbitrarily large $\alpha < \omega$ where there is $\mathcal{M} \models T$ with $A \subseteq M$, $<$ a linear order of A, $|A| = \beth_\alpha$ and for all $a_1 < \ldots < a_n \in A$

$$\mathcal{M} \models \exists \overline{v}\ \sigma(\overline{v}, \overline{a}).$$

We will prove two claims.

Claim Σ is a consistency property.

Claim $\sigma \cup \{\psi\} \in \Sigma$ for all $\sigma \in \Sigma$ and $\psi \in \Gamma$.

Once we have proved these claims we can use the Extended Model Existence Theorem (Exercise 3.7) to conclude there is $\mathcal{M} \models T \cup \Gamma$.

<u>proof of claim 1</u>:

The only tricky case is C4). Suppose $\bigvee_{\psi \in X} \psi \in \sigma \in \Sigma$. Let $\sigma = \sigma(\overline{c}, d_1, \ldots, d_n)$. Then for arbitrarily large α we can find $\mathcal{M}_\alpha$ and $A_\alpha \subset M_\alpha$ of cardinality at least $\beth_{\alpha+n}$ such that for all $a_1, \ldots, a_n \in A_\alpha$

$$\mathcal{M}_\alpha \models \exists \overline{v}\ \sigma(\overline{v}, \overline{a}).$$

Let $f : [A_\alpha]^n \to X$ such that if $f(\overline{a}) = \psi_{\overline{a}}$, then

$$\mathcal{M}_\alpha \models \exists \overline{v}\ (\sigma(\overline{v}, \overline{a}) \wedge \psi_{\overline{a}}(\overline{v}, \overline{a})).$$

Since $\beth_{\alpha+n} \geq \beth^+_{\alpha+n-1}$, $\beth_{\alpha+n} \to (\beth_\alpha)^n_{\aleph_0}$. Thus there is $A'_\alpha \subseteq A_\alpha$ of cardinality $\beth_\alpha$ and ψ_α such that for all $\overline{a} \in A'_\alpha$

$$\mathcal{M}_\alpha \models\ \sigma(\overline{v}, \overline{a}) \wedge \psi_\alpha(\overline{v}, \overline{a}).$$

We can find one $\psi \in X$ such that $\psi = \psi_\alpha$ for aribitarily large $\alpha < \omega$. Then $\sigma \cup \{\psi\} \in \Sigma$.

<u>proof of claim 2</u>:

Suppose $\sigma \in \Sigma$ and θ is

$$\psi(d_{i_1}, \ldots, d_{i_m}) \leftrightarrow \psi(d_{j_1}, \ldots, d_{j_m})$$

where $i_1 < \ldots < i_m$ and $j_1 < \ldots < j_m$. Let $\sigma(\overline{c}, d_1, \ldots, d_n) \in \Sigma$. We must show $\sigma \cup \{\theta\} \in \Sigma$.

There are arbitrarily large $\alpha < \omega$ with $\mathcal{M}_\alpha \models T$ with $A \subseteq M_\alpha$ of cardinality at least $\beth_{\alpha+m}$ such that for all $a_1, \ldots, a_n \in A_\alpha$

$$\mathcal{M}_\alpha \models \exists \overline{v}\ \sigma(\overline{v}, \overline{a}).$$

Let $f : [A_\alpha]^m \to \{0, 1\}$ with $f(\overline{a}) = 1$ if and only if $\mathcal{M}_\alpha \models \psi(\overline{a})$. Since $\beth_{\alpha+m} \to (\beth_\alpha)^m_2$, we can find $A'_\alpha \subseteq A_\alpha$ of cardinality at least $\beth_\alpha$ such that f is constant on $\beth_\alpha$.

This completes the proof.

<u>Alternative Proof</u> This theorem also follows immediately from Theorem 1.14 and the fact the Hanf number for omitting a type in models of first order theory is $\beth_{\omega_1}$. (Theorem 5.2.14 of [9].

Morley's Two Cardinal Theorem

Let $\mathcal{L} = \{U, \ldots\}$ where U is a unary predicate. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is a (κ, λ)-model if $|M| = \kappa$ and $|U^{\mathcal{M}}| = \lambda$.

Theorem 4.8 *Let ϕ be an $\mathcal{L}_{\omega_1,\omega}$-sentence. Suppose for arbitrarily large $\alpha < \omega_1$ there is an infinite κ and $\mathcal{M}$ a $(\beth_\alpha(\kappa), \kappa)$-model of ϕ. Then for all infinite κ there is a $(\kappa, \aleph_0)$-model of ϕ.*

We can extend $\mathcal{L}$ and find a countable fragment $\mathcal{F}$, an $\mathcal{F}$-theory T with built-in-Skolem functions and $\phi \in T$ such that there are aribitrarily large α and infinite κ such that there is a $(\beth_\alpha(\kappa), \kappa)$-model of T.

If $\mathcal{M} \models \phi$ and $I \subseteq M$ is linearly ordered by $<$, we say that I is *indiscernible over U* if for all $\phi(x_1, \ldots, x_n, \overline{u})$ and all $\overline{a} \in U^{\mathcal{M}}$,

$$\mathcal{M} \models \phi(x_1, \ldots, x_n, \overline{a}) \leftrightarrow \phi(y_1, \ldots, y_n, \overline{a})$$

whenever $\overline{x}, \overline{y} \in I$ and $x_1 < \ldots < x_n$, $y_1 < \ldots < y_n$.

Exercise 4.9 Suppose there is $\mathcal{M} \models T$ countable with $I \subseteq M$ an infinite set of indiscernibles over U. Then T has $(\kappa, \aleph_0)$-models for all infinite κ. [Hint: Prove that if f is a Skolem function, $d_1, \ldots, d_n \in I$ and $f(\overline{d}) \in U$, then f is constant in I.]

Add two new countable infinite sets of constant symbols C and D.

Let $\Gamma = \{d_i \neq d_j : i \neq j\} \cup \{\forall \overline{u} \in U\ \theta(d_{i_1}, \ldots, d_{i_m}, \overline{u}) \leftrightarrow \theta(d_{j_1}, \ldots, d_{j_m}.\overline{u}) : \theta(v_1, \ldots, v_m) \in \mathcal{F}, i_1 < \ldots i_m, j_1 < \ldots, j_m\}$.

Let Σ be the set of all finite sets $\sigma(\overline{c}, \overline{d})$ of sentence in $\mathcal{F}$ with finitely many constants from $C \cup D$ such that for aribitrarily large $\alpha < \omega_1$ there is $\mathcal{M} \models T$ and $X \subseteq M$ with $|X| \geq \beth_\alpha(|U^{\mathcal{M}}|)$ such that for all $a_1 < \ldots < a_n \in X$

$$\mathcal{M} \models \exists \overline{v}\ \sigma(\overline{v}, \overline{a}).$$

Exercise 4.10 Prove that Σ is a consistency property.

Exercise 4.11 Show that if $\sigma \in \Sigma$ and $\psi \in \Gamma$, then $\sigma \cup \{\psi\} \in \Sigma$. [Hint: Given $\theta(x_1, \ldots, x_n, \overline{u})$ and $\mathcal{M} \models T$ and $(X, <) \subseteq M$ where $|X| > \beth_{\alpha+n}(|U^{\mathcal{M}}|)$, consider the partition $f : [X]^n \to 2^{|U^{\mathcal{M}}|}$ where $f(x_1, \ldots, x_n) = \{\overline{a} : \mathcal{M} \models \theta(\overline{x}, \overline{a})\}$.]