CLASSIFYING PAIRS OF REAL-CLOSED FIELDS

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE COMMITTEE ON THE GRADUATE DIVISION

OF STANFORD UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

Angus John Macintyre

December 1967

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

(Mathematics)

Approved for the University Committee
on the Graduate Division:

Dean of the Graduate Division

Acknowledgements

I am happy to record here my gratitude to Professor Dana Scott. He suggested the problem to me, provided me with valuable initial information, and guided my subsequent research. I have derived much pleasure from frequent conversations with him, and I thank him for his many kindnesses to me.

My thanks go also to Kenneth Kunen, with whom I have often discussed the metamathematics of field theory, and to whom I often turned for advice. I have derived great benefit from unpublished notes by Professor James Ax. My thinking on the problem has been helped by remarks of Professors Paul Cohen, Andrzej Ehrenfeucht, Solomon Feferman, H. Jerome Keisler, Simon Kochen, Abraham Robinson and Halsey Royden. I have leaned heavily on published work of Ax, Kochen, Ersov, Keisler and Robinson.

I thank the Departments of Mathematics and Philosophy of Stanford University for financial support during my three years at Stanford. In addition, I thank the U.S. Educational Commission in the U.K., for financing my travel to and from the United States.

I thank Martha Kirtley for expert typing of this dissertation, under difficult circumstances, and for various good deeds relating to the preparation and submission of this final version of the dissertation.

I thank Ben Nebres for his kind offer of help with the proofreading. His assistance was invaluable.

In conclusion, I thank my wife, Christina, whose devotion and good humour have been a great encouragement to me.

Table of Contents

CLASSIFYING PAIRS OF REAL-CLOSED FIELDS

by

Angus John Macintyre

Introduction.

The classical work of Steinitz gives a satisfactory classification of the isomorphism-types of algebraically closed fields. It appears desirable to have a similar classification for real-closed fields, but at present we have no such classification. However, Tarski has given a metamathematical classification of real-closed fields, in his theorem that any two real-closed fields are elementarily equivalent with respect to the usual formal system for ordered fields. Tarski proved a corresponding result for algebraically closed fields, namely, that any two algebraically closed fields of the same characteristic are elementarily equivalent with respect to the usual formal system for fields.

A pair (K,L) of fields is a pair (K,L) where K and L are fields and L is a subfield of K. An isomorphism φ of (K_1,L_1) onto (K_2,L_2) is an isomorphism φ of K_1 onto K_2 such that $\varphi[L_1] = L_2$. Using the Steinitz theory, one can readily obtain a satisfactory classification of the isomorphism-types of pairs of algebraically closed fields. On the other hand, it is clearly futile to attempt to classify isomorphism-types of single real-closed fields, without having a classification of isomorphism-types of single real-closed fields.

There is a natural formal system for the theory of pairs of fields, obtained by adjoining a unary predicate-symbol to the formal system for the elementary theory of fields. Similarly, there is a natural formal system for the theory of pairs of ordered fields. Thus we are led to the following questions (i) and (ii).

(i) What are the elementary types of pairs of algebraically closed fields?

(ii) What are the elementary types of pairs of real-closed fields?

A. Robinson answered (i) by proving the following theorem.

<u>Theorem</u>. Suppose K_1, L_1, K_2, L_2 are algebraically closed fields of the same characteristic, such that $L_1 \subseteq K_1$ and $L_2 \subseteq K_2$.

(a) If $K_1 = L_1$ and $K_2 = L_2$, then $(K_1,L_1) \equiv (K_2,L_2)$.

(b) If $K_1 \neq L_1$ and $K_2 \neq L_2$, then $(K_1,L_1) \equiv (K_2,L_2)$.

Dana Scott suggested to the author the second question, and in this work we report on the resulting investigations. It will emerge that there are $2^{\aleph_0}$ elementary types of pairs of real-closed fields, of which only four are known to be axiomatizable. Two of the latter types had previously been identified, by the following theorem.

<u>Theorem</u>. Suppose (K_1,L_1) and (K_2,L_2) are pairs of real-closed fields.

(a) If $K_1 = L_1$ and $K_2 = L_2$, then $(K_1,L_1) \equiv (K_2,L_2)$.

(b) If $K_1 \neq L_1$ and $K_2 \neq L_2$, and L_1 is dense in K_1, and L_2 is dense in K_2, then $(K_1,L_1) \equiv (K_2,L_2)$.

(a) is simply a reformulation of Tarski's Theorem. (b) is a result of A. Robinson, proved in order to solve a problem of Tarski concerning the decidability of the pair $(\mathbb{R}, \tilde{\mathbb{Q}})$, where $\mathbb{R}$ is the reals and $\tilde{\mathbb{Q}}$ is the field of real algebraic numbers. Robinson's proof of (b) is by the method of model-completeness. In this work we prove (b) by the method of ultrapowers. P.J. Cohen has an unpublished proof of (b) which uses the method of quantifier elimination.

We found two new axiomatizable types which are intimately related to (a) and (b) of the above theorem. We partition the class $\mathcal{M}$ of all pairs (K,L) into the class $\mathcal{M}_0$, consisting of pairs (K,L) where L is cofinal in K, and the class $\mathcal{M}_1$, consisting of pairs (K,L) where L is not cofinal in K. It is simple to show that $\mathcal{M}_1$ is non-empty.

Suppose $(K,L) \in \mathcal{M}_1$. We define V^L, the ring of L-bounded elements of K, as the set of all x in K which are bounded above in absolute value by an element of L. V^L is a valuation-ring in K. We define I^L as the ideal of non-units of V^L. We call the elements of I^L the L-infinitesimals. V^L/I^L is a real-closed field, $\mathcal{R}_L$ say. Then L is canonically embedded in $\mathcal{R}_L$, and L is cofinal in $\mathcal{R}_L$. Thus $(\mathcal{R}_L,L)$ is in $\mathcal{M}_0$. Our principal positive result is the following.

<u>Theorem</u>. If (K_1,L_1) and (K_2,L_2) are in $\mathcal{M}_1$, and $(\mathcal{R}_{L_1},L_1) \equiv (\mathcal{R}_{L_2},L_2)$, then $(K_1,L_1) \equiv (K_2,L_2)$.

We obtain this theorem as a consequence of a very important isomorphism theorem of Ax and Kochen. We also have a related result concerning elementary extension rather than elementary equivalence. We use the method of ultrapowers, and are inevitably led to consider pairs (K,L) where K and L are η_1 of cardinality $\aleph_1$. For such pairs we prove some isomorphism theorems which generalize the 1955 result of Erdös, Gillman and Henriksen, namely, that all η_1 real-closed fields of cardinality $\aleph_1$ are isomorphic.

From the principal theorem above we find two new axiomatizable types. The first of those is the type of pairs (K,L) where L is not cofinal in K, and $L = \mathfrak{R}_L$. The second is the type of pairs (K,L) where L is not cofinal in K, and L is a dense, proper subfield of $\mathfrak{R}_L$. Let K be any proper extension of $\mathbb{R}$. Then $(K,\mathbb{R})$ belongs to the first of the new types, and $(K,\mathbb{Q})$ belongs to the second. Our results in this area may be of interest, because they can be construed as completeness and decidability results for certain algebras of infinitesimals.

By our principal theorem, and some elementary results about fields of power-series fields, we can reduce the study of pairs (K,L) to the study of pairs (K,L) where L is cofinal in K. Scott pointed out to us the existence of pairs (K,L) such that L is cofinal in K but not dense in K. Our investigations reveal that there are $2^{\aleph_0}$ elementary types of such pairs.

When L is not dense in K, we study the "gaps" in K, i.e., the intervals of K that do not intersect L. In studying the "gaps", we study the way in which an element of K may be approximated by elements of L. We are led to consider the notion of the closure of L in K. In Section 11, various notions of closure are discussed, and it is shown that, for the most natural notion, the closure of L in K is a real-closed field. A notion of weak density is introduced, and the notion is shown to be weaker than that of density. Weak density is defined in an elementary way, but we show that L is weakly dense in K if and only if L and K have the same group of archimedean classes. When L is weakly dense in K, the "gaps" in K have a fairly simple structure. However, we show that there are at least

$\aleph_0$ elementary types of pairs (K,L) subject to the condition that L is weakly dense in K.

It is easily seen that if $v: K \xrightarrow{\text{onto}} G$ is a valuation, where K is real-closed, then G is a divisible ordered abelian group, or D-group. A converse result is that if K is real-closed and G is a D-group, then $K((t^G))$, the field of formal power series with coefficient in K and exponents in G, is a real-closed field. We make extensive use of the formal power-series construction, and to get the examples in the final section we had to extend some of the classical results.

Our principal negative result is the following.

Theorem. We can interpret within the theory of pairs of real-closed fields the theory of pairs (G,H) of D-groups, subject to the condition that $G \sim H$ is coinitial in G. Within the latter theory we can interpret the theory of an arbitrary linear order.

Since there are $2^{\aleph_0}$ elementary types of linear order, it follows that there are $2^{\aleph_0}$ elementary types of pairs of D-groups, and $2^{\aleph_0}$ elementary types of pairs of real-closed fields.

We conclude by listing some open problems.

Remark on notation. We have tried to be as informal as possible in our presentation. Sometimes we use the same name for different things, and, indeed, sometimes within the one context. For example, "0" may denote, within one context, the zero of several groups or fields. Similar remarks apply to "1", "+" and "<". If we have to distinguish the order relation of a system S we write "$<_S$". Then "$>_S$" denotes the converse relation, and "$\leq_S$" the union of $<_S$ and the identity relation. We do not always distinguish a field from its domain, or from its underlying additive group.

We have assumed in our presentation that the prospective reader will know, from algebra, the material from a textbook such as Jacobson's Volume III, and, from logic, the basic ideas of model theory, including the ultrapower construction. In Section 1 we list, without proof, some classical facts about real-closed fields.

CHAPTER I.

ALGEBRAIC PRELIMINARIES

Section 1. Classical Results on Real-Closed Fields.

Definition 1.1. A field F is formally real if -1 is not a sum of squares in F.

Definition 1.2. F is real-closed if F is formally real and no proper algebraic extension of F is formally real.

Theorem 1.3. If F is an ordered field, the following are equivalent:

a) F is real closed;

b) F(i) is algebraically closed;

c) every positive element of F has a square root in F, and every polynomial of odd degree over F has a root in F.

Theorem 1.4. Every formally real field K can be embedded in a real-closed field F which is algebraic over K. ~~If F_1, F_2 are real-closed algebraic extensions of K, then F_1 is isomorphic to F_2 by an isomorphism that fixes K.~~

Definition 1.5. F is a real closure of K if F is a real-closed algebraic extension of K.

Theorem 1.6.

a) Any real-closed field can be ordered in a unique way.

b) An isomorphism between real-closed fields preserves order.

c) A field can be ordered if and only if it is formally real.

Theorem 1.7. If F is a real-closed field and K is a subfield of F, then K is real closed if and only if K is relatively algebraically closed in F.

Theorem 1.8. If K_1, K_2 are ordered fields with real closures F_1, F_2 respectively, then any order isomorphism of K_1 onto K_2 has a unique extension to an isomorphism of F_1 onto F_2.

Definition 1.9. Let K be an ordered field, and $K(x)$ a pure transcendental ordered extension of K. Then we define $\mathcal{C}(K,x)$ as $\{t \mid t \in K \wedge t <_{K(x)} x\}$

Theorem 1.10. Let F be real closed, and let A be a subset of F such that for all x, y if $x \in A$ and $y < x$ then $y \in A$. Let $F(t)$ be a pure transcendental extension of F. Then we may extend the order on F to an order on $F(t)$ such that $\mathcal{C}(F,t) = A$.

Theorem 1.11. Let F_1, F_2 be real closed, and let φ be an isomorphism of F_1 onto F_2. Let $F_1(t_1)$, $F_2(t_2)$ be pure transcendental ordered extensions of F_1, F_2 respectively. Then φ extends to an order isomorphism of $F_1(t_1)$ onto $F_2(t_2)$, mapping t_1 to t_2, if and only if

$$\varphi[\mathcal{C}(F_1,t_1)] = \mathcal{C}(F_2,t_2) .$$

When φ extends as required above, it extends uniquely.

Remark. When F_1, F_2, t_1, t_2, φ are as above, we say that t_1, t_2 make φ-corresponding cuts in F_1, F_2 if and only if $\varphi[\mathcal{C}(F_1,t_1)] = \mathcal{C}(F_2,t_2)$.

Section 2. **Pairs**

Definition 2.1. A pair of ordered groups is a pair (G,H) where G is an ordered group and H is a subgroup of G with the induced order.

Definition 2.2. A pair of ordered fields is a pair (K,L) where K is an ordered field and L is a subfield of K with the induced order.

Definition 2.3. Let (G_1,H_1), (G_2,G_2) be pairs of ordered groups. Let φ be an order isomorphism of G_1 onto G_2. Then φ is an order isomorphism of (G_1,H_1) onto (G_2,H_2) if and only if $\varphi[H_1] = H_2$. When this holds we write

$$(G_1,H_1) \cong^{\varphi} (G_2,H_2) .$$

Definition 2.4. Let (K_1,L_1), (K_2,L_2) be pairs of ordered fields. Let φ be an order isomorphism of K_1 onto K_2. Then φ is an order isomorphism of (K_1,L_1) onto (K_2,L_2) if and only if $\varphi[L_1] = L_2$. When this holds we write

$$(K_1,L_1) \cong^{\varphi} (K_2,L_2) .$$

Section 3. Density and Cofinality

For the definitions below, $\mathcal{S}$ is an ordered group or an ordered ring, with zero 0, addition +, subtraction - , and order < .

Definition 3.1.

a) If $x \in \mathcal{S}$, $|x|$ is x if $x \geq 0$, and $|x|$ is $-x$ if $x < 0$.

b) A is symmetric in $\mathcal{S}$ if and only if $o \in A$ and $(\forall x)(x \in A \rightarrow -x \in A)$.

c) A is convex in $\mathcal{S}$ if and only if $(\forall x)(\forall y)(\forall z)[(x < z < y \wedge x \in A \wedge y \in A) \rightarrow z \in A]$.

d) A is dense in $\mathcal{S}$ if and only if $(\forall x)(\forall y)[x < y \rightarrow (\exists t)(t \in A \wedge x < t < y)]$.

e) (Assume A symmetric.) A is cofinal in $\mathcal{S}$ if and only if $(\forall x)(\exists y)[\,|x| < |y| \wedge y \in A]$

f) (Assume A symmetric.) A is coinitial in $\mathcal{S}$ if and only if $(\forall x)(\exists y)[x \neq 0 \rightarrow (0 < |y| < |x| \wedge y \in A)]$.

Lemma 3.2. If (K,L) is a pair of ordered fields, then L is cofinal in K if and only if L is coinitial in K.

Proof. The result is trivial, using the fact that, for non-zero x, y in an ordered field K, $|x| < |y|$ if and only if $|y^{-1}| < |x^{-1}|$.

Remark. The result makes essential use of the field axioms, and is not true in general for ordered groups or ordered rings.

Lemma 3.3. If L is an ordered field and K is a real-closure of L then L is cofinal in K.

Proof. Let L be an ordered field and K a real closure of L. K is algebraic over L. Suppose $\alpha \in K$. If $\alpha \in L$, or $|\alpha| \leq 1$, then clearly there is a β in L with $|\alpha| \leq |\beta|$. Now suppose $\alpha \notin L$, and $|\alpha| > 1$. There is an integer n, and $c_1, \dots, c_n$ in L such that $\alpha^n + c_1\alpha^{n-1} + \dots + c_n = 0$. Then $1 + c_1\alpha^{-1} + \dots + c_n\alpha^{-n} = 0$. Then

$$1 = |c_1\alpha^{-1} + \dots + c_n\alpha^{-n}|$$

$$= |\alpha^{-1}| \cdot |c_1 + \dots + c_n\alpha^{-n+1}|$$

$$\leq |\alpha^{-1}| \cdot (|c_1| + |c_2| + \dots + |c_n| .$$

Then $|\alpha| \leq |c_1| + |c_2| + \dots + |c_n|$, and $|c_1| + |c_2| + \dots + |c_n| \in L$. Thus in all cases we have shown that there is a β in L with $|\alpha| < |\beta|$, and this proves the lemma.

Lemma 3.4. There exists a pair (K,L) such that L is an ordered field, K is a real closure of L, and L is not dense in K.

Proof. Let $\mathbb{R}$ be the ordered field of real numbers, and let L be $\mathbb{R}(t)$, a pure transcendental ordered extension of $\mathbb{R}$, ordered so that $\mathcal{C}(\mathbb{R}, t) = \mathbb{R}$. Let K be a real closure of L. Let $\sqrt{t}$ be the positive square root of t in K. Then $\sqrt{t} > r$ for all r in $\mathbb{R}$. By Theorem 1.11, the identity map on $\mathbb{R}$ extends uniquely to an order isomorphism $\varphi: \mathbb{R}(\sqrt{t}) \cong \mathbb{R}(t)$ such that $\varphi(\sqrt{t}) = t$. Suppose there is an f in $\mathbb{R}(t)$ such that $\sqrt{t} < f < 2\sqrt{t}$. Then $\varphi(\sqrt{t}) < \varphi(f) < 2\varphi(\sqrt{t})$. Then $t < \varphi(f) < 2t$. We observe that $\varphi(f) \in \mathbb{R}(t^2)$, and it is easy to check that there is no g in $\mathbb{R}(t^2)$ with $t < g < 2t$. It follows that there is no f in $\mathbb{R}(t)$ such that $\sqrt{t} < f < 2\sqrt{t}$.

Thus L is not dense in K, and the lemma is proved.

Lemma 3.5. If $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair of ordered groups, or a pair of ordered rings, and $\mathcal{S}_2$ is dense in $\mathcal{S}_1$ then $\mathcal{S}_2$ is both cofinal and coinitial in $\mathcal{S}_1$.

Proof. Trivial.

Definition 3.6. If $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair of ordered groups or a pair of ordered rings, then $\mathcal{S}_1 \sim \mathcal{S}_2 = \{x | x \in \mathcal{S}_1 \wedge x \notin \mathcal{S}_2\}$.

Lemma 3.7. If $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair of ordered groups, or a pair of ordered rings, and $\mathcal{S}_1 \sim \mathcal{S}_2$ is not dense in $\mathcal{S}_1$, then $\mathcal{S}_1 \sim \mathcal{S}_2$ is not coinitial in $\mathcal{S}_1$.

Proofs. Let $(\mathcal{S}_1, \mathcal{S}_2)$ be a pair such that $\mathcal{S}_1 \sim \mathcal{S}_2$ is not dense in $\mathcal{S}_1$. Then there exist x, y with $x < y$, and such that all t with $x < t < y$ are in $\mathcal{S}_2$. If $\mathcal{S}_1 \sim \mathcal{S}_2$ is coinitial in $\mathcal{S}_1$, there is a t_1 in $\mathcal{S}_1 \sim \mathcal{S}_2$ with $0 < t_1 < y - x$, and a t_2 with $0 < t_2 < t_1$. Then $x < x + t_2 < x + t_1 < y$, so that $x + t_2$ and $x + t_1$ are in $\mathcal{S}_2$, and there is no u with $x + t_2 < u < x + t_1$ and $u \in \mathcal{S}_1 \sim \mathcal{S}_2$. Then $t_1 - t_2$ is in $\mathcal{S}_2$, and there is no v in $\mathcal{S}_1 \sim \mathcal{S}_2$ with $0 < v < t_1 - t_2$, so that $\mathcal{S}_1 \sim \mathcal{S}_2$ is not coinitial in $\mathcal{S}_1$. This contradiction shows that $\mathcal{S}_1 \sim \mathcal{S}_2$ is not coinitial in $\mathcal{S}_1$, and the lemma is proved.

Lemma 3.8. If (K,L) is a pair of ordered fields, then either $K = L$ or $K \sim L$ is dense in K.

Proof. Suppose (K,L) is a pair of ordered fields. Suppose $K \sim L$ is not dense in K. Then, by Lemma 3.7, $K \sim L$ is not coinitial in K.

Thus there exists an a with $a \neq 0$ such that if $|x| < |a|$ then $x \in L$. In particular $a/2 \in L$. Let x be an arbitrary element of K. Then $|a/2 \cdot (1+|x|)^{-1}| \leq |a/2| < a$. Therefore $a/2 \cdot (1+|x|)^{-1} \in L$. Therefore $(1+|x|)^{-1} \in L$. Therefore $x \in L$. Therefore $K = L$. This proves the lemma.

Section 4. Convexity

Definition 4.1. Let X, Y be ordered sets, and $f: X \to Y$ a map. f is weakly order-preserving if and only if

$$(\forall x_1)(\forall x_2)(x_1 \leq_X x_2 \to f(x_1) \leq_Y f(x_2)).$$

The notion of convexity is important for the following reason. Let $(\mathcal{S}, M)$ be a pair consisting either of an ordered group $\mathcal{S}$ and a subgroup M, or an ordered ring $\mathcal{S}$ and an ideal M in $\mathcal{S}$. Then the quotient system $\mathcal{S}/M$ can be ordered in such a way that the canonical map from $\mathcal{S}$ onto $\mathcal{S}/M$ is weakly order-preserving, if and only if M is convex in $\mathcal{S}$. The relevant order on $\mathcal{S}/M$ is given by the condition: $x + M >_{\mathcal{S}/M} 0 \Longleftrightarrow x \notin M \wedge x >_{\mathcal{S}} 0$.

Lemma 4.2. Let $\mathcal{S}$ be an ordered group or ring, and let M_1, M_2 be convex, symmetric subsets of $\mathcal{S}$. Then $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$.

Proof. Let $\mathcal{S}$, M_1, M_2 be as in the statement of the lemma. Suppose $x \in M_1$, $x \notin M_2$, $y \in M_2$, $y \notin M_1$. By symmetry, $|x| \in M_1$, $|x| \notin M_2$, $|y| \in M_2$, $|y| \notin M_1$. Thus $|x| \neq |y|$. Suppose without loss of generality that $|x| < |y|$. But $0 \in M_2$ and $|y| \in M_2$, and $0 < |x| < |y|$, so by convexity $|x| \in M_2$, contradicting our assumption. It follows that $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$.

Thus the set of convex, symmetric subsets of an ordered group or ring is linearly ordered by inclusion. In particular, the set of convex subgroups of a given ordered group or ring is linearly ordered by inclusion, with smallest element $\{0\}$.

Definition 4.3. Let $\mathcal{S}$ be an ordered group or ring, and $\mathcal{H}$ a collection of convex subgroups of $\mathcal{S}$. Then

$$x \equiv_{\mathcal{H}} y =_{def.} (\forall G \in \mathcal{H})(x \in G \leftrightarrow y \in G)$$

$$x \ll_{\mathcal{H}} y =_{def.} (\forall G \in \mathcal{H})(x \in G \rightarrow y \in G) .$$

Lemma 4.4.

(a) $(\forall x)(\forall y)(x \equiv_{\mathcal{H}} y \longleftrightarrow (x \ll_{\mathcal{H}} y \wedge y \ll_{\mathcal{H}} x))$.

(b) $\equiv_{\mathcal{H}}$ is an equivalence relation.

(c) $(\forall x)(\forall y)(x \ll_{\mathcal{H}} y \vee y \ll_{\mathcal{H}} x)$.

(d) Let $\mathcal{H}(x)$ be the equivalence class of x with respect to $\equiv_{\mathcal{H}}$, and let $\Delta = \{\mathcal{H}(x) | x \in \mathcal{S}\}$. Then Δ is (linearly) ordered by the condition:

$$\mathcal{H}(x) \leq \mathcal{H}(y) =_{def.} x \ll_{\mathcal{H}} y .$$

(e) $\mathcal{H}(nx) = \mathcal{H}(x)$, if $n \in Z - \{0\}$.

(f) $\mathcal{H}(x+y) \geq \min (\mathcal{H}(x), \mathcal{H}(y))$, for all x, y.

(g) $\mathcal{H}(0) \geq \mathcal{H}(x)$ for all x, and if $\{0\} \in \mathcal{H}$ then $\mathcal{H}(x) = \mathcal{H}(0)$ if and only if $x = 0$.

(h) $(\forall x)(\forall y)(|x| \leq |y| \rightarrow \mathcal{H}(y) \leq \mathcal{H}(x))$.

Proof. (a) and (b) are trivial.

(c) Suppose $|x| \leq |y|$. Then if $G \in \mathcal{H}$, and $y \in G$, $x \in G$ by convexity of G. Thus $y \ll_{\mathcal{H}} x$. Since $|x| \leq |y|$, or $|y| \leq |x|$, (c) follows.

(d) In view of (a), (b), (c) we need only show

(d') $$(\forall x_1, x_2, y_1, y_2)[(x_1 \equiv_{\mathcal{H}} x_2 \wedge y_1 \equiv_{\mathcal{H}} y_2 \wedge x_1 \ll_{\mathcal{H}} y_1) \rightarrow x_2 \ll_{\mathcal{H}} y_2] .$$

So, suppose $x_1 \equiv_{\mathcal{H}} x_2$ and $y_1 \equiv_{\mathcal{H}} y_2$ and $x_1 \ll_{\mathcal{H}} y_1$. Suppose $G \in \mathcal{H}$, and $y_2 \notin G$. Then $y_1 \notin G$. Then $x_1 \notin G$. Then $x_2 \notin G$. Thus, if $G \in \mathcal{H}$ and $x_2 \in G$. Thus $x_2 \ll_{\mathcal{H}} y_2$. This proves (d).

(e) Clearly for all x, and for n in Z, $\mathcal{H}(x) \leq \mathcal{H}(nx)$. Conversely, $|x| < |n| \cdot |x|$ for $n \in Z - \{0\}$, and $|n| \cdot |x| = |nx|$. Therefore, if $G \in \mathcal{H}$ and $nx \in G$, where $n \in Z - \{0\}$, then $x \in G$. Thus $\mathcal{H}(nx) \leq \mathcal{H}(x)$ if $n \in Z - \{0\}$.

(f) Suppose without loss of generality that $\mathcal{H}(x) \leq \mathcal{H}(y)$. Suppose $G \in \mathcal{H}$, and $x \in G$. Then $y \in G$, so that $x + y \in G$. Thus $x \ll_{\mathcal{H}} x + y$. Therefore, $\mathcal{H}(x+y) \geq \mathcal{H}(x) = \min(\mathcal{H}(x), \mathcal{H}(y))$.

(g) Trivial.

(h) See the proof of (c).

Section 5. Valuations

We assume familiarity with the concept of a valuation on an arbitrary field. In this section we develop the concept of a convex valuation, both for ordered groups and ordered fields.

Definition 5.1. Let $\mathcal{S}$ be an ordered group or an ordered ring. Then $\mathcal{S}^*$ is $\mathcal{S} - \{0\}$.

Definition 5.2. Let $\mathcal{S}$ be an ordered group, and Λ an ordered set $\langle A, <_A \rangle$. A function v from $\mathcal{S}^*$ onto Λ is a convex group-valuation of $\mathcal{S}$ onto Λ if and only if

(a) $(\forall x)(\forall y)(v(x+y) \geq \min(v(x), v(y)))$, and

(b) $(\forall x)(\forall y)(|x| \geq |y| \rightarrow v(x) \leq v(y))$

Lemma 5.3. Let v be a convex group-valuation of $\mathcal{S}$ onto Λ. Then

(i) $v(-x) = v(x)$ for all x in $\mathcal{S}^*$,

(ii) $v(x+y) = \min(v(x), v(y))$ if $v(x) \neq v(y)$, for all x, y in $\mathcal{S}^*$ and

(iii) $v(nx) = v(x)$ for all x in $\mathcal{S}^*$ and n in $Z - \{0\}$.

Proof.

(i) By (b) of 5.2, if $|x| = |y|$ then $v(x) = v(y)$. Therefore $v(-x) = v(x)$.

(ii) Suppose $v(x) \neq v(y)$, and without loss of generality $v(x) > v(y)$. Then

$$v(y) = v(x+y-x) \geq \min(v(x+y) , v(-x))$$

$$= \min(v(x+y), v(x)) .$$

Since $v(x) > v(y)$ it follows that $v(y) \geq v(x+y)$. On the other hand, $v(x+y) \geq \min(v(x),v(y)) = v(y)$. Therefore, $v(x+y) = v(y) = \min(v(x),v(y))$.

(iii) Suppose $n \in Z$, and $n > 0$. By induction, using 5.2(a), we get $v(nx) \geq v(x)$. Conversely, $|nx| \geq |x|$, so, by (b), $v(nx) \leq v(x)$. Then $v(nx) = v(x)$. The general result follows, using (i).

5.4. If $\mathcal{S}$ is an ordered group, and $\mathcal{H}$ a collection of convex subgroups of $\mathcal{S}$ as in 4.3 and 4.4, we get an associated valuation as follows. Let Λ be $\{\mathcal{H}(x) | x \in \mathcal{S}^*\}$, ordered as in 4.4(d). Then by 4.4(f) and 4.4(h), the map $v: \mathcal{S}^* \to \Lambda$, where $v(x) = \mathcal{H}(x)$ for all x in $\mathcal{S}^*$, is a convex group-valuation of $\mathcal{S}$ onto Λ.

5.5. Let $\mathcal{S}$ be an ordered field, and $\mathcal{S}_+$ the underlying ordered additive group. We have a special interest in the convex group-valuations $v: \mathcal{S}_+^* \to \Lambda$ which satisfy the condition (Mult) below.

(Mult): $(\forall x_1,x_2,y_1,y_2)[(v(x_1) \geq v(x_2) \wedge v(y_1) \geq v(y_2))$

$$\to v(x_1y_1) \geq v(x_2y_2)] .$$

<u>Lemma 5.6</u>. If (Mult) holds we can define an operation of addition on Λ, whereby Λ becomes an ordered group. Addition is defined by:

$$v(x) + v(y) = v(xy) \quad \text{for all } x, y$$

<u>Proof</u>. Suppose (Mult) holds. By two applications of (Mult) we see that if $v(x_1) = v(x_2)$, and $v(y_1) = v(y_2)$, then $v(x_1y_1) = v(x_2y_2)$.

It follows that the proposed addition is well-defined. It is clearly commutative and associative. $v(1)$ is a neutral element, and for any x in $\mathcal{S}^*$ $v(x) + v(x^{-1}) = v(1)$. This proves that the proposed addition gives Λ a group structure.

By the convexity of v, we have for any x in $\mathcal{S}^*$ that $v(x^{-1}) \geq v(1) \geq v(x)$ or $v(x) \geq v(1) \geq v(x^{-1})$, according to whether $|x| \geq 1$, or $|x| \leq 1$. Thus $v(x) \geq v(1)$ or $-v(x) \geq v(1)$, and if both hold then $v(x) = v(1)$. If $v(x) \geq v(1)$ and $v(y) \geq v(1)$ then by (Mult) $v(xy) \geq v(1)$, whence $v(x) + v(y) \geq v(1)$. This completes the proof.

Definition 5.7. Let $\mathcal{S}$ be an ordered field, and G an ordered group. A function v from $\mathcal{S}^*$ onto G is a convex valuation of $\mathcal{S}$ onto G if and only if

(a) $(\forall x)(\forall y)(v(x+y) \geq \min(v(x),v(y)))$,

(b) $(\forall x)(\forall y)(|x| \geq |y| \rightarrow v(x) \leq v(y))$, and

(c) $(\forall x)(\forall y)(v(xy) = v(x) + v(y))$.

Remark. Clearly a convex valuation on an ordered field is simply a special kind of valuation on a field. We will use the notions of valuation-ring, residue-class field, and place, without further explanation.

Lemma 5.8. Let $\mathcal{S}$ be an ordered field, and v a convex valuation of $\mathcal{S}$ onto the ordered group G. Let V be the valuation-ring of v, and I the maximal ideal of non-units of V. Let $\pi\colon V \rightarrow V/I$ be the canonical place. Then V is convex in $\mathcal{S}$, I is convex in V,

V/I has the natural induced order, and π is weakly order-preserving.

Proof. Assume the hypothesis of the lemma. Suppose x and y are in V, and $x < t < y$ with $t \neq 0$. Then $|t| \leq \max(|x|,|y|)$. Suppose without loss of generality that $|t| \leq |x|$. By convexity, $v(t) \geq v(x) \geq 0$. Therefore $t \in V$. Therefore V is convex in $\mathcal{S}$.

Suppose x and y are in I, and $x < t < y$. If $t = 0$ then $t \in I$. We have $|t| \leq \max(|x|,|y|)$. Suppose $t \neq 0$ and without loss of generality $|t| \leq |x|$. We have $|x^{-1}| \leq |t^{-1}|$. If $t^{-1} \in V$ it follows by convexity of V that $x^{-1} \in V$, whence $x \notin I$. Therefore, $t^{-1} \notin V$, and $t \in I$. Therefore I is convex in $\mathcal{S}$.

The rest of the lemma is trivial, by earlier remarks about convexity.

Henceforward, when we talk of the residue-class field of a convex valuation we mean the residue-class field with the natural ordering.

Definition 5.9. Let $\mathcal{S}$ be an ordered field. A subset T of $\mathcal{S}$ is quasi-archimedean if and only if

(a) $2 \in T$,

(b) $(\forall x)(x \in T \rightarrow |x| \in T)$, and

(c) $(\forall x)(\forall y)[x \in T \wedge y \in T) \rightarrow xy \in T]$

Definition 5.10. Let $\mathcal{S}$ be an ordered field and T a quasi-archimedean subset of $\mathcal{S}$. $\mathcal{H}^T$ is the collection of convex (additive) subgroups G of $\mathcal{S}$, such that $(\forall x)(x \in G \rightarrow (\forall t)(t \in T \rightarrow t \cdot x \in G))$.

Definition 5.11. v^T is the group-valuation on $\mathcal{S}^*$ associated with $\mathcal{H}^T$.

Lemma 5.12. If $x, y \in \mathcal{S}^*$ then $v^T(x) \geq v^T(y)$ if and only if there is a t in T such that $|x| \leq t \cdot |y|$.

Proof. By the definition of $\mathcal{K}^T$, if there is a t in T with $|x| \leq t \cdot |y|$ then $v^T(x) \geq v^T(y)$.

Conversely, suppose $v^T(x) \geq v^T(y)$. Let G_y be $\{z | (\exists\, t \in T)(|z| \leq t|y|)\}$. Since $2 \in T$, $y \in G_y$. Clearly G_y is convex and symmetric. If $|z_1| \leq t_1 \cdot |y|$, and $|z_2| \leq t_2 \cdot |y|$, where $t_1, t_2 \in T$, then $|z_1 \pm z_2| \leq 2\max(t_1,t_2) \cdot |y|$. But $2\max(t_1,t_2) \in T$, by 5.9(a) and 5.9(c). Therefore, G_y is a convex subgroup of $\mathcal{S}_+$. By 5.9(c), $G_y \in \mathcal{K}^T$. Since $y \in G_y$, $x \in G_y$, so there is a t in T with $|x| \leq t \cdot |y|$. This proves the lemma.

Lemma 5.13. v^T satisfies (Mult).

Proof. Suppose $v^T(x_1) \geq v^T(x_2)$, and $v^T(y_1) \geq v^T(y_2)$. Then by 5.12 there are t, u in T such that $|x_1| \leq t \cdot |x_2|$, and $|y_1| \leq u \cdot |y_2|$. Then $|x_1y_1| \leq tu \cdot |x_2y_2|$, and $tu \in T$ by 5.9(c). Therefore by 5.12 $v^T(x_1y_1) \geq v^T(x_2y_2)$. This proves the lemma.

By 5.13 and 5.6 there is a natural structure of ordered group on $v^T[\mathcal{S}^*]$, and we henceforward construe v^T as a convex valuation on the field $\mathcal{S}$.

We note that Z is quasi-archimedean in any ordered field $\mathcal{S}$. v^Z is the valuation onto the so-called group of archimedean classes of $\mathcal{S}$, and $\mathcal{K}^Z$ is the collection of all convex subgroups of $\mathcal{S}$. For x, y in $\mathcal{S}^*$, $v^Z(x) = v^Z(y)$ if and only if there are integers m, n such that $|x| \leq |m|\ |y|$ and $|y| \leq |n| \cdot |x|$.

To any pair (K,L) of ordered fields we associate a convex valuation on K, as follows. It is clear that L is quasi-archimedean in K. Then v^L is a convex valuation on K, henceforward known as the canonical valuation of (K,L). Let V^L be the valuation-ring of v^L, and I^L the maximal ideal of non-units of V^L.

<u>Lemma 5.14</u>. Let (K,L) be a pair of ordered fields, and v^L the canonical valuation. Then

(a) $V^L = \{x | (\exists\, \ell)(\ell \in L \wedge |x| \leq |\ell|)\}$

(b) $I^L = \{x | (\forall\, \ell)[(\ell \in L \in \ell \neq 0) \rightarrow |x| < |\ell|]\}$.

<u>Proof</u>. Let K, L, v^L be as in the statement of the lemma.

(a) $x \in V^L \iff (x = 0 \vee v^L(x) \geq 0)$

$\iff (x = 0 \vee v^L(x) \geq v^L(1))$

$\iff (\exists\, \ell)(\ell \in L \wedge |x| \leq |\ell|$

using 5.12.

(b) This follows easily from (a).

<u>Definition 5.15</u>. Let (K,L) be a pair of ordered fields, and x, y elements of K.

(a) x is L-bounded if and only if $x \in V^L$.

(b) x is L-infinitesimal if and only if $x \in I^L$.

(c) x is L-infinitesimally close to y if and only if $x - y$ is L-infinitesimal.

Section 6. Convex Valuations on Real-Closed Fields

Definition 6.1. Let (K,L) be a pair of ordered fields, and v^L the canonical valuation. Let $\mathcal{R}(L)$ be the (ordered) residue-class field V^L/I^L, and let π^L be the canonical place from V^L onto $\mathcal{R}(L)$.

Recall that if M is a field $\dot{M}^*$ is $M - \{0\}$.

Lemma 6.2. Suppose (K,L) is a pair of ordered fields, and $x \in L^*$. Then $v^L(x) = 0$.

Proof. Trivial, by 5.14(a).

Thus $L \subseteq V^L$, and π^L injects L into $\mathcal{R}(L)$.

Lemma 6.3. $\pi^L[L]$ is cofinal in $\mathcal{R}(L)$.

Proof. π^L is weakly order-preserving from V^L onto $\mathcal{R}(L)$. Suppose $x \in V^L$. Then by 5.14(a), there is an ℓ in L with $|x| < |\ell|$. But then $|\pi^L(x)| \leq |\pi^L(\ell)|$. This proves the lemma.

Definition 6.4. Let K be an ordered field and v a convex valuation on K. Let M be a subfield of K. Then v is trivial on M if and only if for each x in M^* $v(x) = 0$.

Lemma 6.5. Let (K,L) be a pair of ordered fields. Then there exist subjields M of K maximal with respect to the property that L is a subfield of M and v^L is trivial on M.

Proof. The proof is obvious, by 6.2 and Zorn's lemma.

Definition 6.6. Let (K,L) be a pair of ordered fields, and let M be a subfield of K maximal with respect to the property that L is a subfield of M and v^L is trivial on M. Then M is said to be L-maximal in K.

Lemma 6.7. Let (K,L) be a pair of ordered fields, and M an L-maximal subfield of K. Then $\mathcal{R}(L)$ is algebraic over $\pi^L[M]$.

Proof. Let (K,L) be a pair of ordered fields, and let M be an L-maximal subfield of K. Then π^L is a monomorphism on M. Suppose $x \in V^L$ and $\pi^L(x)$ is transcendental over $\pi^L[M]$.

Suppose $t \in M[x]$. Then there is an integer m and element $\mu_0, \ldots, \mu_m$ in M such that $t = \sum_{r=0}^{m} \mu_r x^r$. Then $\pi^L(t) = \sum_{r=0}^{m} \pi^L(\mu_r) \cdot (\pi^L(x))^r$. But then since $\pi^L(x)$ is transcendental over $\pi^L[M]$ it follows that $\pi^L(t) = 0$ if and only if each $\pi^L(\mu_r) = 0$. Since π^L is a monomorphism on M it follows that $\pi^L(t) = 0$ if and only if $t = 0$. It follows that $v^L(t) = 0$ if $t \in M[x] - \{0\}$, and then it follows that v^L is trivial on $M(x)$.

Thus if $\pi^L(x)$ is transcendental over $\pi^L[M]$ then M is not L-maximal. The lemma follows.

Lemma 6.8. Let (K,L) be a pair of ordered fields, with K real closed. Let M be an L-maximal subfield of K. Then M is real closed, and $\pi^L[M] = \mathcal{R}(L)$.

Proof. Let (K,L) be a pair of ordered fields, with K real closed. Let M be an L-maximal subfield of K. Then π^L is a monomorphism on M. Let M_1 be the relative algebraic closure of M in K. Then the proof of 3.3 shows that M is cofinal in M_1. Since $M \subset V^L$ it follows by 5.14(a) that $M_1 \subset V^L$, and so since M_1 is a field v^L is trivial on M_1. By the maximality of M, $M_1 = M$. Therefore M is relatively algebraically closed in K, and so by 1.7 M is real closed. But then $\pi^L[M]$ is real closed, since π^L is a monomorphism on M. By 6.7, $\mathcal{R}(L)$ is an ordered algebraic extension of $\pi^L[M]$. Therefore $\mathcal{R}(L) = \pi^L[M]$.

Corollary 6.9. Let (K,L) be a pair of ordered fields, with K real closed. Then $\mathcal{R}(L)$ is real closed.

Proof. By 6.5, there are L-maximal subfields of K. The result now follows by 6.8.

Lemma 6.10. Let K be an ordered field, and v a convex valuation on K, with valuation-ring V. Let M be a maximal subfield of K on which v is trivial. Then $V = V^M$, the valuation-ring of v^M

Proof. Let K, v, V, M be as in the statement of the lemma. Let π be the place associated with v. Then examination of the proof of 6.7 shows that $\pi[V]$ is algebraic over $\pi[M]$. Then, by 3.3, $\pi[M]$ is cofinal in $\pi[V]$.

Suppose $x \in V$. By the preceding remarks, there is an m in M such that $|\pi(x)| \le |\pi(m)| = \pi(|m|)$. Now, if $|x| > |m| + 1$,

$$\pi(|x|) \ge \pi(|m| + 1) = \pi(|m|) + 1 > \pi(|m|) .$$

Therefore $|x| \le |m| + 1$, and since $|m| + 1 \in M$ it follows that $x \in V^M$, using 5.14(a). Therefore $V \subseteq V^M$.

Conversely, suppose $x \in V^M$. Then by 5.14(a) there is an in M with $|x| \leq |m|$. By convexity of v it follows that $x \in V$, since $m \in V$. Therefore $V^M \subseteq V$. This concludes the proof.

Section 7. Extension Theory for Convex Valuations

Definition 7.1. Let $\langle F_i, v_i, G_i\rangle$, $(i = 1,2)$, be triples consisting of an ordered field F_i, an ordered group G_i, and a convex valuation v_i of F_i onto G_i. An analytic isomorphism of $\langle F_1, v_1, G_1\rangle$ onto $\langle F_2, v_2, G_2\rangle$ is a pair of maps $\langle \varphi, \psi\rangle$ such that

a) φ is an order-isomorphism of F_1 onto F_2,

b) ψ is an order-isomorphism of G_1 onto G_2,

c) $\psi v_1 = v_2 \varphi$.

Let (K,L) be a pair of ordered fields. In Section 5 we defined v^L as a convex valuation of K onto a certain group of equivalence classes of the relation $\equiv_{\mathcal{K} L}$. For our purposes the valuation-ring of v^L is more important than the value-group, and we shall sometimes identify v^L with arbitrary convex valuations on K which have valuation-ring V^L.

When we defined v^L we had a specific pair (K,L) in mind, and our notation for v^L omitted reference to K. In this section and the next we encounter situations where we have to consider simultaneously pairs (K,L) and (K_1,L). Then v_K^L and $v_{K_1}^L$ are the respective valuations defined as in Section 5 for the pairs (K,L) and (K_1,L).

Lemma 7.2. Let (K,L), (K_1,L) be pairs of ordered fields, where K_1 is an extension of K. Then $v_{K_1}^L$ extends v_K^L.

Proof. Let (K,L), (K_1,L) be as in the statement of the lemma. Suppose $x, y \in K^*$ and $v_K^L(x) = v_K^L(y)$. Then by 5.14(a) there are ℓ_1, ℓ_2 in L such that $0 < |\ell_1| \leq |x/y| \leq |\ell_2|$. But then, applying 5.14(a) again, $v^L_{K_1}(x) = v^L_{K_1}(y)$. This proves the lemma.

Theorem 7.3. Let K be an ordered field, and v a convex valuation of K onto an ordered group. Let K_1 be a real-closure of K. Then, up to an analytic isomorphism $\langle\varphi,\psi\rangle$ where φ is the identity map of K_1, v has a unique extension to a convex valuation v_1 on K_1.

Proof. Let K, v, K_1 be as in the statement of the theorem.

Let M be a maximal subfield of K on which v is trivial. Then, by 6.10, $v = v^M_K$. By 7.2, $v^M_{K_1}$ is a convex extension of v to K_1.

Let v_1 be any convex valuation on K_1 with v_1 extending v. Let $V^M_{K_1}$ be the valuation-ring of $v^M_{K_1}$. Let V_1 be the valuation-ring of v_1. We will show that $V_1 = V^M_{K_1}$, and this will establish the uniqueness part of the theorem, and will conclude the proof.

Let π_1 be the place associated with v_1. Suppose $x \in V_1$. Now x is algebraic over K. Thus there is an integer n, and $c_o, \ldots, c_n$ in K such that $c_n \neq 0$ and $\sum_{r=o}^{n} c_r \cdot x^r = 0$. Let r_o be the least r such that $|c_{r_o}|$ is maximal in the set $\{|c_o|, \ldots, |c_n|\}$. Let c'_r be c_r/c_{r_o}. Then $|c'_r| \leq 1$, so $v_1(c'_r) \geq 0$ for $0 \leq r \leq n$. Further, $c'_{r_o} = 1$, so that $v_1(c'_{r_o}) = 0)$ and $\pi_1(c'_{r_o}) = 1$. We have

$$\sum_{r=o}^{n} \pi_1(c'_r) \cdot (\pi_1(x))^r = 0 ,$$

so that $\pi_1(x)$ is algebraic over $\pi_1[V^M_K]$. Thus $\pi_1(x)$ is algebraic over $\pi^M[V^M_K]$. By 6.7, $\pi^M[V^M_K]$ is algebraic over $\pi^M[M]$.

It follows that $\pi_1[V_1]$ is algebraic over $\pi^M[M]$. On the other hand, let M_1 be the relative algebraic closure of M in K_1. Then, by 1.7, M_1 is a real-closure of M. By 3.3, M is cofinal in M_1. Since $M \subset V_K^L$, and v_1 is a convex extension of v_K^L, it follows that $M_1 \subset V_1$, and since M_1 is a field v_1 is trivial on M_1. But then π_1 is a monomorphism on M_1. Thus $\pi_1[M_1]$ is real closed. Since $\pi_1[M_1] \subseteq \pi_1[V_1]$, and $\pi_1[V_1]$ is an ordered algebraic extension of $\pi^M[M]$, it follows that $\pi_1[V_1] = \pi_1[M_1]$.

We can now show that $V_1 = \{x \mid (\exists m)(m \in M \wedge |x| \leq |m|\}$.

Suppose $x \in V_1$. Since $\pi_1[V_1] = \pi_1[M_1]$, it follows that there is an m_1 in M_1 such that $v_1(x-m_1) > 0$. By convexity, $|x-m_1| \leq 1$. Therefore $|x| \leq |m_1| + 1$. Since M is cofinal in M_1, there is an m in M with $|m_1| + 1 < |m|$. Therefore $|x| < |m|$.

On the other hand, suppose $x \in K_1^*$ and $|x| \leq |m|$ for some m in M. Then $m \neq 0$, and $v_1(x) \geq v_1(m)$, by convexity. But $v_1(m) = v(m) = 0$. Therefore $v_1(x) \geq 0$, so $x \in V_1$.

Therefore $V_1 = V_{K_1}^M$, by 5.14(a), and the theorem is proved.

Lemma 7.4. Let v be a convex valuation of an ordered field K. Let π be the place associated with v, and suppose M is a subfield of K on which v is trivial, and such that π is an isomorphism on M. Let K_1 be a real closure of K, and let v_1 be the unique convex extension of v to K_1. Let M_1 be the relative algebraic closure of M in K_1. Then v_1 is trivial on M_1, and π_1, the place of v_1, is an isomorphism on M_1.

Proof. The result is true even if we require only that M be a maximal subfield of K on which v is trivial. The proof is contained in the proof of 7.3.

Lemma 7.5. Suppose K is a real-closed field and M is a M-maximal subfield of K. Suppose F is a real-closed subfield of K such that $\pi^M[V^M \cap F] = \pi^M[M \cap F]$. Suppose $m \in M - F$. Let F_1 be $F(m)$. Then

$$\pi^M[V^M \cap F_1] = \pi^M[M \cap F_1] .$$

Proof. Assume the hypotheses of the lemma. Suppose $x \in F_1$. Then there are integers n, m and $c_o, \ldots, c_n, d_o, \ldots, d_m$ in F such that $x = \sum_{i=o}^{n} c_i \cdot m^i / \sum_{j=o}^{m} d_j m^j$. Let i_o be the least i such that $|c_i|$ is maximal in $\{|c_o|, \ldots, |c_n|\}$. Let j_o be the least j such that $|d_j|$ is maximal in $\{|d_o|, \ldots, |d_m|\}$. For $0 \le i \le n$, let c_i' be c_i/c_{i_o}. For $0 \le j \le m$, let d_j' be d_j/d_{j_o}. Then $c_{i_o}' = 1$, $d_{j_o}' = 1$, and so $\pi^M(c_{i_o}') = 1$, $\pi^M(d_{j_o}') = 1$. Then

$$x = c_{i_o}/d_{j_o} \sum_{i=o}^{n} c_i' \cdot m^i / \sum_{j=o}^{m} d_j' \cdot m^j .$$

Now, by an argument like that in 6.8, $\pi^M[V^M \cap F]$ is real closed, and so $\pi^M(m)$ is transcendental over $\pi^M[V^M \cap F]$. Since $|c_i'| \le 1$, and $|d_j'| \le 1$, it follows that $c_i' \in V^M$ and $d_j' \in V^M$. Then

$$\pi^M\left(\sum_{i=o}^{n} c_i' \cdot m^i \right) = \sum_{i=o}^{n} \pi^M(c_i') \cdot (\pi^M(m))^i \neq 0$$

$$\pi^M\left(\sum_{i=o}^{m} d_j' \cdot m^j \right) = \sum_{j=o}^{m} \pi^M(d_j') \cdot (\pi^M(m))^j \neq 0 .$$

Thus $x \in V^M$ if and only if $c_{i_o}/d_{j_o} \in V^M$. Thus if $x \in V^M$,

$$\pi^M(x) = \pi^M(c_{i_o}/d_{j_o}) \cdot \frac{\sum_{i=o}^{n} \pi^M(c_i') \cdot (\pi^M(m))^i}{\sum_{j=o}^{m} \pi^M(d_j') \cdot (\pi^M(m))^j} .$$

Now, by hypothesis there are $m_i (o \leq i \leq n)$, $\mu_j (0 \leq j \leq m)$, and v in $M \cap F$ such that

$$\pi^M(v) = \pi^M(c_{i_o}/d_{j_o}) ,$$

$$\pi^M(m_i) = \pi^M(c_i') ,$$

$$\pi^M(\mu_j) = \pi^M(d_j') .$$

But then $\pi^M(x) = \pi^M\left(v \cdot \frac{\sum_{i=o}^{n} m_i \cdot m^i}{\sum_{j=o}^{m} \mu_o \cdot m^j}\right) \in \pi^M[M \cap F_1]$. The lemma follows easily.

Section 8. D-Groups

Definition 8.1. Let G be an abelian group. G is divisible (resp. uniquely divisible) if and only if for each x in G and positive integer n there is a y (resp. a unique y) in G such that $ny = x$.

Lemma 8.2. If G is divisible and torsion free then G is uniquely divisible.

Proof. Trivial.

8.3. If G is uniquely divisible, and $x \in G$, and n is a positive integer, then $\frac{1}{n} \cdot x$ is defined as the unique y such that $ny = x$.

Definition 8.4. Let G be an ordered abelian group. Then G is a D-group if and only if G is divisible.

8.5. It is clear that a D-group is torsion free, and so is uniquely divisible. It is clear that a D-group is densely ordered, and clear also that the additive group of an ordered field is a D-group.

For us the chief importance of D-groups comes from the following lemma.

Lemma 8.6. Let K be a field in which every element has an n^{th} root, for each positive integer n. Let v be a valuation of K onto G. Then G is a D-group.

Proof. Assume the hypothesis of the lemma. Suppose $g \in G$, and n is a positive integer. Select x in K with $v(x) = g$, and then select y in K with $y^n = x$. Then $v(y^n) = nv(y)$. Thus there is an h in G with $nh = g$. Since g was arbitrary, G is divisible, and so, G is D-group.

Corollary. Suppose K is either algebraically closed or real closed. Let v be a valuation of K onto G. Then G is a D-group.

Proof. Clear.

8.7. We now know that if K is real closed and v is a convex valuation of K onto G, with residue-class field F, then F is real closed and G is a D-group. In Section 10, we discuss the formal-power-series construction which, given F real closed and G a D-group, will yield real-closed K and convex $v: K \to G$ with residue-class field F.

Section 9. Hahn Groups

9.1. In the notes at the end we refer the reader to some of the literature on Hahn groups. For our purposes the deeper facts of the theory, e.g., Hahn's Embedding Theorem, are not relevant.

Let $\langle \Lambda, < \rangle$ be a linearly ordered set. Let $\{\langle H_\lambda, +_\lambda, 0_\lambda, <_\lambda \rangle\}_{\lambda \in \Lambda}$ be an indexed family of ordered groups. Let $\langle X, +, 0 \rangle$ be the complete direct product $\prod_{\lambda \in \Lambda} \langle H_\lambda, +_\lambda, 0_\lambda \rangle$. X is, of course, $\prod_\lambda H_\lambda$, the cartesian product of the H_λ. If $f \in X$, we define supp(f), the support of f, as $\{\lambda | f(\lambda) \neq 0_\lambda\}$. Let W be the subset of X consisting of those f for which supp(f) is well ordered by $<$. It is a simple, known result that W is a subgroup of X. If $f \in W^*$, we define $v(f)$ as the least λ such that $f(\lambda) \neq 0_\lambda$. We define an order $<_W$ on W by:

$$f <_W g =_{def} f \neq g \wedge 0 <_{v(f-g)} (g(v(f-g)) - f(v(f-g))) .$$

Again, it is a simple known result that $\langle W, +, 0, <_W \rangle$ is an ordered group, and v is a convex valuation of W onto Λ. Adopting the convention of identifying $\langle H_\lambda, +_\lambda, 0_\lambda, <_\lambda \rangle$ with H_λ, for each λ in Λ, let $\Gamma_{\lambda \in \Lambda} H_\lambda$ be $\langle W, +, 0, <_W \rangle$.

9.2. If A is a subset of Λ, well ordered by $<$, let ord(A) be the ordinal of A with respect to $<$. If $f \in W$, we define $\|f\|$ as ord(supp(f)).

Lemma 9.3. a) if A and B are well-ordered subsets of Λ, then $A \cup B$ is well ordered.

(b) If A and B are well ordered subsets of Λ, then $\text{ord}(A \cup B) \leq (1+\text{ord}(B)) \cdot (1+\text{ord}(A)+1)$.

(c) If f, g are in W, then $\text{supp}(f+g) \subseteq \text{supp}(f) \cup \text{supp}(g)$.

(d) If f, g are in W then $\|f + g\| \leq (1+\|g\|) \cdot (1+\|f\|+1)$.

Proof.

(a) This is obvious.

(b) Suppose A and B are well-ordered subsets of Λ. Let λ be $\text{ord}(A)$, and let μ be $\text{ord}(B)$. Thus, in the order $<$, A may be well-ordered as $\{a_\tau\}_{\tau<\lambda}$. Let B_{-1} be the subset of B consisting of those b in B such that $b < a_\tau$ for all τ. Let B_∞ be the subset of B consisting of those b in B such that $a_\tau < b$ for all τ. For $\tau < \lambda$ we define B_τ as the subset of B consisting of those b in B such that $a_\tau < b < a_{\tau+1}$, unless $\tau + 1 = \lambda$, when we define B_τ as B_∞. Then clearly

$$\begin{aligned}\text{ord}(A \cup B) &\leq \text{ord}(B_{-1}) + \sum_{\tau+1<\lambda} (1+ \text{ord}(B_\tau)) + \text{ord}(B_\infty) \\ &\leq \mu + (1+\mu)\lambda + \mu \\ &\leq (1+\mu) + (1+\mu)\lambda + (1+\mu) \\ &= (1+\mu)(1+\lambda+1) .\end{aligned}$$

This proves (b).

(c) is clear.

(d) This follows from (b) and (c).

9.4. Let σ be an ordinal closed under addition and multiplication. Let $W^{(\sigma)}$ be the subset of W consisting of those f in W for which $\|f\| < \sigma$. Then, by 9.3, $\langle W^{(\sigma)}, +, 0, <_W \cap (W^{(\sigma)})^2\rangle$ is an ordered group, for which we use the notation $\Gamma^{(\sigma)}_{\lambda\in\Lambda} H_\lambda$.

Lemma 9.5. $\Gamma_{\lambda\in\Lambda} H_\lambda$ and $\Gamma^{(\sigma)}_{\lambda\in\Lambda} H_\lambda$ are D-groups if and only if all the H_λ are D-groups.

Proof. Trivial.

Section 10. Formal Power Series

10.1. In this section we adopt a procedure like that of Section 9. We will give a brief explanation of the classical construction, and list without proof certain basic results pertaining to this construction. From these results we will obtain estimates analogous to 9.3(d).

To begin with, let $\langle F, +, \cdot, 0, 1\rangle$ be an arbitrary commutative ring. (Later we restrict F to be a field, and still later to be an ordered field). Let $\langle G, +_G, 0_G, <_G\rangle$ be an ordered group. We will define $F((t^G))$, the ring of formal power series with coefficients in F and exponents in G.

Firstly, let X be the abelian group $\prod_{g\in G} \langle F, +, 0\rangle$. If $f \in X$, we define $\mathrm{supp}(f)$ as $\{g | f(g) \neq 0\}$. Let W be the subset of X consisting of those f in X for which $\mathrm{supp}(f)$ is well-ordered by $<_G$. Then, as in 9, W is a subgroup of X.

It is convenient to construe W as the group of formal power series $\sum_g f(g) \cdot t^g$ with well-ordered support. t is a formal symbol, and addition is defined coordinatewise.

We are going to define on W a multiplication $\cdot$, which will make W into a ring. (No confusion should result from the use of the same symbols for addition and multiplication in F and W.) If $\sum_g f_1(g) \cdot t^g$ and $\sum_g f_2(g) \cdot t^g)$ are in W, we define

$$\left(\sum_g f_1(g) \cdot t^g\right) \cdot \left(\sum_g f_2(g) \cdot t^g\right)$$

as

$$\sum_g \left(\sum_h f_1(h) \cdot f_2(g-h) \right) \cdot t^g .$$

It is well known that if f_1 and f_2 are in W and $g \in G$, then the sum $\sum_h f_1(h) \cdot f_2(g-h)$ is really a finite sum. To show that W is closed under $\cdot$, we need essentially the following fact.

Fact. Let A, B be subsets of G, well ordered with respect to $<_G$. Define $\Sigma^{(2)}(A,B)$ as the set of all sums $\alpha +_G \beta$, where $\alpha \in A$ and $\beta \in B$. Then $\Sigma^{(2)}(A,B)$ is well ordered by $<_G$.

The above fact is rather clear, and in the notes we give references to the literature.

The unit element of W is $1 \cdot t^o$. It is a routine exercise to check that W is a ring under $+$ and $\cdot$. F is canonically embedded in W, by the map $x \rightsquigarrow x \cdot t^o$.

10.2. If A is a subset of G, well ordered by $<_G$, let $\mathrm{ord}(A)$ be the ordinal of A with respect to $<_G$. If $f \in W$, we define $\|f\|$ as $\mathrm{ord}(\mathrm{supp}(f))$. Just as in 9.3(d) we have the estimate $\|f + g\| \leq (1+\|g\|) \cdot (1+\|f\| + 1)$, for f and g in W. We are now going to obtain an estimate for $\|fg\|$.

10.3. We define a binary relation R between pairs of ordinals by:

$$(\lambda_o,\mu_o)R(\lambda_1,\mu_1) =_{\mathrm{def}} \lambda_o \leq \lambda_1 \wedge \mu_o \leq \mu_1 \wedge (\lambda_o,\mu_o) \neq (\lambda_1,\mu_1) ,$$

for all ordinals λ_o, μ_o, λ_1, μ_1. It is a simple matter to check that R is a partial order, and R is well founded. By means of transfinite recursion on R we will define a binary function P from ordinals to ordinals.

Definition of P. $P(0,0) = 1$. If $(0,0)R(\lambda,\mu)$, then

$$P(\lambda,\mu) = [\sup_{(\lambda_1,\mu_1)R(\lambda,\mu)} (1+P(\lambda_1,\mu_1)+1)^2] + 1 .$$

Lemma 10.4. Suppose A and B are well-ordered subsets of G, with respect to $<_G$. Let λ_A, λ_B be ord(A), ord(B) respectively. Then

$$\operatorname{ord}(\sum{}^{(2)}(A,B)) \leq P(\lambda_A,\lambda_B) .$$

Proof. We prove the result by transfinite induction on (λ_A,λ_B) with respect to R. If $(\lambda_A,\lambda_B) = (0,0)$, the result is clear. Suppose now that we have proved the result for all pairs A_1, B_1 for which $(\operatorname{ord}(A_1), \operatorname{ord}(B_1))R(\lambda_A,\lambda_B)$. Well-order A, with respect to $<_G$, as $\{\alpha_\lambda\}_{\lambda<\lambda_A}$, and well-order B, with respect to $<_G$, as $\{b_\mu\}_{\mu<\lambda_B}$. For each $\tau \leq \lambda_A$, let A^τ be $\{\alpha_\lambda\}_{\lambda<\tau}$. For each $\varepsilon \leq \lambda_B$, let B^ε be $\{b_\mu\}_{\mu<\varepsilon}$. Let τ_o, ε_o be fixed ordinals such that $\tau_o < \lambda_A$ and $\varepsilon_o < \lambda_B$. Then, if $\alpha_\tau +_G \beta_\varepsilon <_G \alpha_{\tau_o} +_G \beta_{\varepsilon_o}$, either $\tau < \tau_o$ or $\varepsilon < \varepsilon_o$. Thus the set of predecessors, in $\Sigma^{(2)}(A,B)$, of $\alpha_{\tau_o} +_G \beta_{\varepsilon_o}$ is a subset of $\Sigma^{(2)}(A^{\tau_o},B) \cup \Sigma^{(2)}(A,B^{\varepsilon_o})$. Now $(\operatorname{ord}(A^{\tau_o}), \operatorname{ord}(B)) = (\tau_o,\lambda_B)$, and $(\tau_o,\lambda_B)R(\lambda_A,\lambda_B)$. Similarly, $(\operatorname{ord}(A), \operatorname{ord}(B^{\varepsilon_o}))R(\lambda_A,\lambda_B)$. It follows by induction that $\operatorname{ord}(\Sigma^{(2)}(A^{\tau_o},B)) \leq P(\tau_o,\lambda_B)$, and $\operatorname{ord}(\Sigma^{(2)}(A,B^{\varepsilon_o})) \leq P(\lambda_A,\varepsilon_o)$. It then follows, using 9.3(b), that $\operatorname{ord}(\Sigma^{(2)}(A^{\tau_o},B) \cup \Sigma^{(2)}(A,B^{\varepsilon_o})) \leq (1+P(\tau_o,\lambda_B)+1) \cdot (1+P(\lambda_A,\varepsilon_o)+1)$ $\leq \sup_{(\lambda_1,\mu_1)R(\lambda_A,\lambda_B)} (1+P(\lambda_1,\mu_1)+1)^2$. It follows that every initial segment of $\Sigma^{(2)}(A,B)$ has ordinal

$$\leq [\sup_{(\lambda_1,\mu_1)R(\lambda_A,\lambda_B)} (1+P(\lambda_1,\mu_1)+1)^2] + 1$$

Therefore, by definition of P,

$$\operatorname{ord}(\Sigma^{(2)}(A,B)) \leq P(\lambda_A,\lambda_B) .$$

This completes the proof.

Corollary 10.5. Suppose f and g are in W. Then $\|fg\| \leq P(\|f\|,\|g\|)$.

Proof. This is clear, from 10.4, and the fact that

$$\operatorname{supp}(fg) \leq \Sigma^{(2)}(\operatorname{supp}(f), \ \operatorname{supp}(g)) .$$

10.6. Suppose now that σ is an ordinal closed under addition, multiplication, and P. Then, by the above, $W^{(\sigma)}$, the set of those f in W such that $\|f\| < \sigma$, is a subring of W. Let $F((t^G))$ be W, and let $F((t^G))_\sigma$ be $W^{(\sigma)}$.

10.7. If n is a positive integer, and $A_1, \ldots, A_n$ are subsets of G, we define $\Sigma^{(n)}(A_1,\ldots,A_n)$ as the set of all sums $g_1 +_G \cdots +_G g_n$ where $g_1 \in A_1, \ldots, g_n \in A_n$. This definition generalizes that of $\Sigma^{(2)}$, given in 10.1. It is easy to see that if $A_1, \ldots, A_n$ are well ordered by $<_G$ then so is $\Sigma^{(n)}(A_1,\ldots,A_n)$.

If A is a subset of G we define $n \cdot A$ as $\Sigma^{(n)}(A,\ldots,A)$. We define $\infty \cdot A$ as $\bigcup_{n=1}^{\infty} n \cdot A$. Even if A is well ordered by $<_G$, $\infty \cdot A$ need not be well ordered by $<_G$. However, it is a classical fact that if A is a well-ordered positive subset of G (i.e., every member of A is positive) then $\infty \cdot A$ is well ordered.

Another important classical fact about positive well-ordered A is that if $g \in \infty \cdot A$, then there are only finitely many n such that $g \in n \cdot A$. We refer to the literature for proofs of these results.

10.8. On the basis of the preceding subsection, one may establish the classical fact that if F is a field then $F((t^G))$ is a field. For, suppose F is a field, and x is a non-zero element of $F((t^G))$. Then, by an easy argument, x is of the form $a \cdot t^{\gamma} \cdot (1+h)$ where $a \in F^*$, $\gamma \in G$, and h has positive support. Since $a \cdot t^{\gamma}$ is certainly invertible, we have our result as soon as we show that $1 + h$ is invertible. But from the results of 10.7 one sees that $\sum_{n=0}^{\infty} (-1)^n h^n$ is a well-defined element of $F((t^G))$, and then by a familiar formal argument $\sum_{n=0}^{\infty} (-1)^n h^n$ is the inverse of $1 + h$.

We are going to show that a similar argument may be carried out in $F((t^G))_\sigma$ for certain σ. For h of positive support, we will obtain an estimate for $\|\sum_{n=0}^{\infty} (-1)^n h^n\|$ in terms of $\|h\|$.

Definition 10.9. For $1 \leq n < \omega$ we define functions p_n from ordinals to ordinals by

$$p_1(\lambda) = \lambda \quad \text{for all } \lambda ;$$

$$p_2(\lambda) = P(\lambda,\lambda) \quad \text{for all } \lambda ;$$

$$p_{n+2}(\lambda) = P(\lambda, p_{n+1}(\lambda)) \quad \text{for all } \lambda .$$

One may easily check that the p_n are non-decreasing functions.

Lemma 10.10. If $1 \leq n < \omega$, and A is a subset of G well ordered by $<_G$, then $\operatorname{ord}(n \cdot A) \leq p_n(\operatorname{ord}(A))$.

Proof. This is clear from 10.4 and 10.9.

Lemma 10.11. If $A_1, \ldots, A_k$ are finitely many well-ordered subsets of G, then $\operatorname{ord}(\bigcup_{m=1}^{k} A_m) \leq \prod_{m=1}^{k} (1+\operatorname{ord}(A_m)+1)$.

Proof. The result is trivial for $k = 1$. We prove the general result by induction on k. Suppose we have proved that

$$\operatorname{ord}\Big(\bigcup_{m=1}^{k} A_m\Big) \leq \prod_{m=1}^{k} (1+\operatorname{ord}(A_m)+1) .$$

Now, by 9.3(b),

$$\operatorname{ord}\Big(\bigcup_{m=1}^{k+1} A_m\Big) \leq \Big(1+\operatorname{ord}\Big(\bigcup_{m=1}^{k} A_m\Big)\Big) \cdot (1+\operatorname{ord}(A_{k+1})+1) .$$

We distinguish two cases.

Case (i). $\operatorname{ord}\Big(\bigcup_{m=1}^{k} A_m\Big)$ infinite.

Then

$$1 + \operatorname{ord}\Big(\bigcup_{m=1}^{k} A_m\Big) = \operatorname{ord}\Big(\bigcup_{m=1}^{k} A_m\Big) ,$$

so

$$\begin{aligned}\operatorname{ord}\Big(\bigcup_{m=1}^{k+1} A_m\Big) &\leq \operatorname{ord}\Big(\bigcup_{m=1}^{k} A_m\Big) \cdot (1+\operatorname{ord}(A_{k+1})+1) \\ &\leq \prod_{m=1}^{k} (1+\operatorname{ord}(A_m)+1) \cdot (1+\operatorname{ord}(A_{k+1})+1) \\ &= \prod_{m=1}^{k+1} (1+\operatorname{ord}(A_m)+1) .\end{aligned}$$

<u>Case (ii)</u>. $\operatorname{ord}\Big(\bigcup_{m=1}^{k} A_m\Big)$ finite.

Then for $1 \leq m \leq k$, $\text{ord}(A_m)$ is finite, and then clearly

$$\text{ord}(\bigcup_{m=1}^{k} A_m) \leq \sum_{m=1}^{k} \text{ord}(A_m) .$$

Then

$$1 + \text{ord}(\bigcup_{m=1}^{k} A_m) \leq 1 + \sum_{m=1}^{k} \text{ord}(A_m)$$

$$\leq \prod_{m=1}^{k} (1+\text{ord}(A_m)+1) .$$

Therefore

$$\text{ord}(\bigcup_{m=1}^{k+1} A_m) \leq \prod_{m=1}^{k+1} (1+\text{ord}(A_m)+1) .$$

This gives the inductive step of the proof, and we are through.

Definition 10.12. For all ordinals λ,

$$p_\omega(\lambda) = \lim_{k\to\omega} \prod_{m=1}^{k} (1+p_m(\lambda)+1) .$$

10.13. As a first step in estimating $\text{ord}(\infty\cdot A)$, when A is a positive well-ordered subset of G, we discuss the special case in which A satisfies (Arch): If α_0 is the least element of A, and a is any element of A, then there is an integer n such that $a \leq n \cdot \alpha_0$.

Lemma 10.14. If A is a positive well-ordered subset of G satisfying (Arch), then $\text{ord}(\bigcup_{n=1}^{\infty} n\cdot A) \leq p_\omega(\text{ord}(A))$.

Proof. If $A = \emptyset$, the result is trivial. Suppose now A is non-empty and satisfies the conditions of the lemma. Let x be an arbitrary element of $\bigcup_{n=1}^{\infty} n \cdot A$. Then, since x is a finite sum of elements of A, and A satisfies (Arch), there are positive integers k such that $x \leq k \cdot \alpha_o$. Let k_o be the least such k. Suppose now $y \in \bigcup_{n=1}^{\infty} n \cdot A$, and $y \leq_G x$. Then there is an integer n, and elements $a_1, \ldots, a_n$ in A such that $y = a_1 +_G \cdots +_G a_n$. Then $a_1 +_G \cdots +_G a_n \leq x \leq k \cdot \alpha_o$. But clearly $n\alpha_o \leq a_1 +_G \cdots +_G a_n$, and, since α_o is positive, $n \leq k_o$. Therefore $y \in \bigcup_{n=1}^{k_o} n \cdot A$. Thus $\{y | y \in \infty \cdot A \wedge y \leq_G x\}$ is a subset of $\bigcup_{n=1}^{k_o} n \cdot A$. Now, by 10.10 and 10.11,

$$\begin{aligned} \text{ord}\left(\bigcup_{n=1}^{k_o} n \cdot A\right) &\leq \prod_{n=1}^{k_o} (1+p_n(\text{ord}(A))+1) \\ &\leq p_\omega(\text{ord}(A)) . \end{aligned}$$

Since x was arbitrary, the result follows.

10.15. After Lemma 5.13 we mentioned the convex valuation v^Z on an ordered field $\mathcal{S}$. We observed that, for x,y in $\mathcal{S}^*$, $v^Z(x) = v^Z(y)$ if and only if there are positive integers m, n such that $|x| \leq m|y|$ and $|y| \leq n|x|$. It is clear that by the latter property we may define a convex group-valuation on an arbitrary ordered group. We denote the resulting group valuation by "v^Z" also, without risk of confusion.

It is easily seen that (Arch) holds for a well-ordered positive set A if and only if for all a in A $v^Z(a) = v^Z(\alpha_o)$, where α_o is the least element of A.

For the general case, let A be a positive well-ordered subset of G. If $x \in A$, we define A_x as $\{y | y \in A \wedge v^Z(y) = v^Z(x)\}$. Since v^Z is convex, we may introduce a linear order $<$ on $\{A_x\}_{x \in A}$ by:

$$A_x < A_y =_{def} x <_G y \wedge A_x \neq A_y .$$

Since A is well ordered by $<_G$ it is easily seen that $\{A_x\}_{x \in A}$ is well ordered by $<$. We define ε_A, the index of A, as the ordinal of $\{A_x\}_{x \in A}$ under $<$. Clearly $\varepsilon_A \leq \text{ord}(A)$. We will shortly define a binary function q from ordinals to ordinals and show that $\text{ord}(\infty \cdot A) \leq q(\text{ord}(A), \varepsilon_A)$.

Let x be an element of $\infty \cdot A$. Then there is an m, and $a_1, \ldots, a_m$ in A such that $x = \sum_{j=1}^{m} {}_G\, a_j$. Then, since the a's are positive, $v^Z(x) = \min_{1<j<m} v^Z(a_j)$. Define $A^{(x)}$ as $\{y \in A | v^Z(y) \geq v^Z(x)\}$. On the basis of the preceding few lines, the following facts are evident.

<u>Fact 1</u>. $\{u | u \in \infty \cdot A \wedge u \leq_G x\}$ is a subset of $\infty \cdot A^{(x)}$.

<u>Fact 2</u>. $A^{(x)} = (\bigcup_{A_y < A_x} A_y) \cup A_x$.

<u>Fact 3</u>. $\infty \cdot A^{(x)} = \infty \cdot (\bigcup_{A_y < A_x} A_y) \cup \infty \cdot A_x \cup \Sigma^{(2)}(\infty \cdot (\bigcup_{A_y < A_x} A_y), \infty \cdot A_x)$.

<u>Fact 4</u>. A_x satisfies (Arch).

<u>Fact 5</u>. The index of $\bigcup_{A_y < A_x} A_y$ is less than the index of $A^{(x)}$.

Motivated by the above facts, we can now define q.

Definition 10.16. For arbitrary ordinals λ:

(i) $q(\lambda,0) = 0$;

(ii) $q(\lambda,1) = p_\omega(\lambda)$;

(iii) if $\sigma > 1$,

$$q(\lambda,\sigma) = (1+\sup_{\mu<\sigma} q(\lambda,\mu)+1) \cdot (1+p_\omega(\lambda)+1) \cdot (1+P(\sup_{\mu<\sigma} q(\lambda,\mu),p_\omega(\lambda))+1) .$$

Lemma 10.17. If A is a positive, well ordered subset of G, then

$$\text{ord}(\infty\cdot A) \leq q(\text{ord}(A),\varepsilon_A) .$$

Proof. We observe first that the functions P and q are non-decreasing in both arguments.

The case $\varepsilon_A = 1$ is covered by 10.14. We prove the general result by induction on the index of A. Suppose we have proved the result for all B with $\varepsilon_B < \varepsilon_A$. Let x be an arbitrary element of $\infty \cdot A$. Then by Fact 1, $\{u | u \in \infty \cdot A \wedge u \leq_G x\}$ is a subset of $\infty \cdot A^{(x)}$. Let B be $\bigcup_{A_y<A_x} A_y$. Then $\varepsilon_B < \varepsilon_A$, and $A^{(x)} = B \cup A_x$, by Fact 2. By Fact 3, $\infty \cdot A^{(x)}$ is a subset of $\infty \cdot B \cup \infty \cdot A_x \cup \Sigma^{(2)}(\infty \cdot B, \infty \cdot A_x)$. Let λ be $\text{ord}(\infty \cdot B)$, and let μ be $\text{ord}(\infty \cdot A_x)$. By Fact 5, and the induction hypothesis,

$$\begin{aligned}\text{ord}(\infty \cdot B) &\leq q(\text{ord}(B),\varepsilon_B)\\ &\leq \sup_{\tau<\varepsilon_A} q(\text{ord}(B),\tau) .\end{aligned}$$

By Fact 4 and 10.14,

$$\operatorname{ord}(\infty \cdot A_x) \le q(\operatorname{ord}(A_x), \varepsilon_{A_x})$$

$$= p_\omega(\operatorname{ord}(A_x))$$

$$\le p_\omega(\operatorname{ord}(A)) .$$

By 10.4,

$$\operatorname{ord}(\Sigma^{(2)}(\infty \cdot B, \infty \cdot A_x)) \le P(\lambda,\mu)$$

$$\le P(\sup_{\tau<\varepsilon_A} q(\operatorname{ord}(A),\tau), p_\omega(\operatorname{ord}(A))) .$$

Now, by 10.11 and the definition of q, we see easily that

$$\operatorname{ord}(\infty \cdot B \cup \infty \cdot A_x \cup \Sigma^{(2)}(\infty \cdot B, \infty \cdot A_x)) \le q(\operatorname{ord}(A),\varepsilon_A) .$$

It follows that $\operatorname{ord}(\infty \cdot A^{(x)}) \le q(\operatorname{ord}(A),\varepsilon_A)$. Since x was arbitrary, the result follows, using Fact 1.

Corollary 10.18. If A is a positive, well-ordered subset of G, then

$$\operatorname{ord}(\infty \cdot A) \le q(\operatorname{ord}(A), \operatorname{ord}(A)) .$$

Proof. Clear, since $\varepsilon_A \le \operatorname{ord}(A)$.

Lemma 10.19. Suppose F is a field and σ is closed under q (as well as $+$, $\cdot$, and P). Then $F((t^G))_\sigma$ is a field.

Proof. If $h \in F((t^G))_\sigma$ and h has positive support, then by 10.18

$$\left\| \sum_{n=0}^{\infty} (-1)^n h^n \right\| \le 1 + q(\|h\|, \|h\|) .$$

The result now follows from the remarks in 10.7.

10.20. Henceforward, in this section we assume F is a field and σ is closed under $+$, $\cdot$, P and q. Our final aim in this section is to show that if F is real closed and G is a D-group, then $F((t^G))_\sigma$ is real closed.

Henceforward, in this section let K be $F((t^G))_\sigma$. We observe that we have the obvious canonical valuation v from K onto G, with residue-class field F. Let V be the valuation ring of v. Let π be the canonical place of V onto F. If $f(x)$ is a polynomial $\Sigma\, a_n x^n$ with coefficients in V, then we define $\overline{f}(x)$ as the polynomial $\Sigma\, \pi(a_n) \cdot x^n$ in $F[x]$.

Lemma 10.21. (Hensel's Lemma). Suppose $f(x) \in V[x]$ is monic, and suppose $\overline{f} = \varphi \cdot \psi$, where φ, ψ are relatively prime elements of $F[x]$. Then there are g, h in $V[x]$ such that $f = g \cdot h$, $\overline{g} = \varphi$, $\overline{h} = \psi$, $\deg(g) = \deg(\varphi)$, and $\deg(h) = \deg(\psi)$.

Proof. Let f, φ, ψ be as in the statement of the lemma. Let m be $\deg(\varphi)$, and let n be $\deg(\psi)$. Then $m + n = \deg(f) = N$, say. $f(x)$ is of the form $x^N + \lambda_1 \cdot x^{N-1} + \dots + \lambda_N$, where, for $1 \leq r \leq N$, λ_r is a formal power series in t, with non-negative support, and ordinal less than σ. Let B be the union of the supports of the λ_n, and 0. Then, by a previously mentioned classical result, $\infty \cdot B$ is a well-ordered semigroup of G. Moreover by 10.17 and the closure properties of σ, $\infty \cdot B$ has ordinal less than σ. Let A be $\infty \cdot B$. Then for $1 \leq r \leq N$, and $\alpha \in A$, there are $a_{r,\alpha}$ in F such that

$$\lambda_r = \sum_{\alpha\in A} a_{r,\alpha} \cdot t^\alpha .$$

We now arrange $f(x)$ in powers of t as $\sum_{\alpha\in A} f_\alpha(x)\cdot t^\alpha$, where $f_\alpha(x) = \overline{f}(x)$ if $\alpha = 0$, and $f_\alpha(x) = \sum_{r=1}^{N} a_{r,\alpha} x^{N-r}$, if $\alpha \neq 0$. Clearly if $\alpha \neq 0$, then f_α is of degree at most $(N-1)$. We define $g_o(x)$ as $\varphi(x)$, and $h_o(x)$ as $\psi(x)$. Then $f_o(x) = g_o(x)\cdot h_o(x)$, and g_o and h_o are relatively prime.

We are going to express the required $g(x)$ and $h(x)$ as $\sum_{\alpha\in A} g_\alpha(x)\cdot t^\alpha$ and $\sum_{\alpha\in A} h_\alpha(x)\cdot t^\alpha$ respectively, where the g_α and h_α are in $F[x]$.

Proceeding formally, we see that

$$\left(\sum_{\alpha\in A} g_\alpha(x)\cdot t^\alpha\right)\cdot\left(\sum_{\alpha\in A} h_\alpha(x)\cdot t^\alpha\right) = \sum_{\alpha\in A} f_\alpha(x)\cdot t^\alpha$$

if and only the following equations hold:

$$\sum_{\substack{\alpha_1,\alpha_2\in A,\\ \alpha_1+_G\alpha_2=\alpha}} g_{\alpha_1}(x)\cdot h_{\alpha_2}(x) = f_\alpha(x) \qquad (\alpha\in A).$$

We are going to demand that $\deg(g_\alpha) \leq m-1$, $\deg(h_\alpha) \leq n-1$, if $\alpha \neq 0$.

Since A is well ordered, for any α in A there are only finitely many α_1, α_2 in A such that $\alpha = \alpha_1 +_G \alpha_2$. Also, since A contains no negative elements, if $\alpha, \alpha_1, \alpha_2 \in A$ and $\alpha = \alpha_1 +_G \alpha_2$ then $\alpha_1 \leq_G \alpha$ and $\alpha_2 \leq_G \alpha$.

We have already solved that one of the above equations which corresponds to $\alpha = 0$. We have chosen g_o and h_o so that $g_o h_o = f_o$.

We define A^* as $A \sim \{0\}$. For $\alpha \in A^*$, the corresponding equation may be reformulated as

$$g_o(x)h_\alpha(x) + g_\alpha(x) \cdot h_o(x) = f_\alpha(x) - \sum_{\substack{\alpha_1,\alpha_2 \in A^* \\ \alpha_1 +_G \alpha_2 = \alpha}} g_{\alpha_1}(x) \cdot h_{\alpha_2}(x) .$$

We observe that if $\alpha_1, \alpha_2 \in A^*$ and $\alpha_1 +_G \alpha_2 = \alpha$ then $\alpha_1 <_G \alpha$ and $\alpha_2 <_G \alpha$. Suppose we have defined g_β, h_β for all β in A with $\beta <_G \alpha$ in such a way that

i) $g_\beta, h_\beta \in F[x]$;

ii) $\deg(g_\beta) \leq m - 1$, $\deg(h_\beta) \leq n - 1$;

iii) $\sum_{\substack{\alpha_1,\alpha_2 \in A \\ \alpha_1 +_G \alpha_2 = \beta}} g_{\alpha_1}(x) \cdot h_{\alpha_2}(x). \quad = f_\alpha(x)$

We define $B_\alpha(x)$ as $f_\alpha(x) - \sum_{\substack{\alpha_1,\alpha_2 \in A^* \\ \alpha_1 +_G \alpha_2 = \alpha}} g_{\alpha_1}(x) \cdot h_{\alpha_2}(x)$. Then $B_\alpha(x) \in F[x]$, and if $\alpha \in A^*$, $\deg B_\alpha \leq N - 1$. Now since g_o, h_o are relatively prime and respectively of degrees m, n, it follows by elementary algebra [53, Ch. 2, §14] that there are g_α, h_α in $F[x]$, with $\deg(g_\alpha) \leq m - 1$, $\deg(h_\alpha) \leq n - 1$, such that

$$g_o(x)h_\alpha(x) + g_\alpha(x)h_o(x) = B_\alpha(x) .$$

But then

$$\sum_{\substack{\alpha_1,\alpha_2\in A\\ \alpha_1+_G\alpha_2=\alpha}} g_{\alpha_1}(x)\cdot h_{\alpha_2}(x) = f_\alpha(x) .$$

It follows, using the fact that A is well ordered by $<_G$, that we may inductively define g_α, h_α $(\alpha\in A)$ in $F[x]$ such that:

(1) $g_0 = \varphi$, $h_0 = \psi$;

(2) if $\alpha \neq 0$, $\deg(g_\alpha) \leq m - 1$, $\deg(h_\alpha) \leq n - 1$;

(3) $(\sum_{\alpha\in A} g_\alpha(x) \cdot t^\alpha) \cdot (\sum_{\alpha\in A} h_\alpha(x) \cdot t^\alpha) = \sum_{\alpha\in A} f_\alpha(x) \cdot t^\alpha$.

We now define $g(x)$ as $\sum_{\alpha\in A} g_\alpha(x) \cdot t^\alpha$, and $h(x)$ as $\sum_{\alpha\in A} h_\alpha(x) \cdot t^\alpha$. Then clearly $g(x)$ and $h(x)$ are in $V[x]$, and $g(x)h(x) = f(x)$. Clearly also $\deg(g) = m$, and $\deg(h) = n$. Finally, $\bar{g} = g_0 = \varphi$, and $\bar{h} = h_0 = \psi$. This proves the lemma.

The following lemma is in the literature, and we later give references.

Lemma 10.22. Let U be a field and H an ordered group, and v a valuation of U onto G, with residue-class field T. Suppose

(i) $\langle U, v, H\rangle$ satisfies Hensel's lemma,

(ii) H is a D-group.

(iii) T is algebraically closed of characteristic 0.

Then U is algebraically closed.

Corollary 10.23. If F_1 is algebraically closed of characteristic 0, and G is a D-group, then $F_1((t^G))_\sigma$ is algebraically closed.

Proof. Clear, by 10.21 and 10.22.

10.24. From now to the end of the section, suppose F is an ordered field, with order $<_F$. Then we may order $F((t^G))_\sigma$ so that the canonical valuation v onto G is convex. Suppose $x \in F((t^G))_\sigma$ and $x \neq 0$. Let g be $v(x)$. As usual, let π be the canonical place associated with v. π is a map onto F. Then we say x is positive if and only if $0 <_F \pi(t^{-g} \cdot x)$.

Recall we write K for $F((t^G))_\sigma$. We order K by:

$$x <_K y =_{def} y - x \text{ is positive .}$$

It is easy to check that $<_K$ is an order on K, and that with respect to this order $v: K \to G$ is convex.

Lemma 10.25. If F is real closed, and G is a D-group, then $F((t^G))_\sigma$ is real closed.

Proof. Let F, G be as in the statement of the lemma. Let i be a square root of -1, and let F_1 be $F(i)$. Then by 1.3(b), F_1 is algebraically closed of characteristic 0. As usual, let K be $F((t^G))_\sigma$. Then since F has an order, K has an order, by 10.24. Now it is a simple observation that $K(i) = F_1((t^G))_\sigma$. But by Corollary 10.23, $F_1((t^G))_\sigma$ is algebraically closed. Therefore $K(i)$ is algebraically closed. Therefore by 1.3(b), K is real closed. This concludes the proof.

10.26. We leave it as an exercise for the reader to prove that if σ is an uncountable cardinal then σ is closed under $+$, $\cdot$, P and q. From this observation we get, by selecting σ as a cardinal greater than the maximum of $\aleph_0$ and the cardinal of G, that if F is real closed and G is a D-group then $F((t^G))$ is real closed. (This result is classical).

A final important fact, again left as an exercise, is that there are countable ordinals closed under $+$, $\cdot$, P and g. We will later make an application of this fact.

Section 11.

Closure

In this section (K,L) is a fixed pair of ordered fields. If x, ε are in K, we define $Nbd(x,\varepsilon)$ as $\{y \mid y \in K \wedge |x - y| < \varepsilon\}$. The sets $Nbd(x,\varepsilon)$ form a base for a topology $\mathcal{J}$ on K. It is clear that $+$ and $\cdot$ are continuous in the product topology. The function f defined by $f(0) = 0$, $f(x) = x^{-1}$ otherwise, is continuous except at 0.

Let $\bar{L}$ be the closure of the set L in K. $\bar{L}$ consists of those elements k of K such that for each $\varepsilon > 0$ in K $Nbd(k,\varepsilon) \cap L \neq 0$. It is clear that $k \in \bar{L}$ if and only if there is an ordinal λ, and a series $\{\ell_\mu\}_{\mu<\lambda}$ from L such that if ε is any positive element of K, then there is a $\mu_0 < \lambda$ such that if $\mu > \mu_0$, then $|k - \ell_\mu| < \varepsilon$.

Lemma 11.1. L is discrete in K with respect to $\mathcal{J}$ if and only if L is not cofinal in K.

Proof. This is clear.

Let $cf(K)$ be the least ordinal λ such that there exists a well-ordered monotone increasing series $\{x_\tau\}_{\tau<\lambda}$ which is cofinal in K.

Lemma 11.2. If $x \in \bar{L}$, there is a net $\{\ell_\tau\}_{\tau<cf(K)}$ of elements of L such that $\{\ell_\tau\}_{\tau<cf(K)}$ converges to x in $\mathcal{J}$.

Proof. If L is not cofinal in K, then $L = \bar{L}$ by 11.1, and the result is trivial.

Suppose L is cofinal in K, and $x \in \bar{L}$. Then there is an ordinal δ and a net $\{\ell_\mu\}_{\mu<\delta}$ from L such that $\ell_\mu \neq x$ for all $\mu < \delta$, and $\{|\ell_\mu - x|\}_{\mu<\delta}$ is monotone decreasing to 0. Then $\{|\ell_\mu - x|^{-1}\}_{\mu<\delta}$

is monotone increasing and cofinal in K so that $\delta \geq cf(K)$. Now let $\{y_\tau\}_{\tau<cf(K)}$ be a monotone increasing series cofinal in K. We define $\mu(\tau)$, for $\tau < cf(K)$, as the least $\mu < \delta$ such that $|\ell_\mu - x|^{-1} > y_\tau$. Then clearly $\{\ell_{\mu(\tau)}\}_{\tau<cf(K)}$ converges to x. This completes the proof.

Lemma 11.3. $\overline{L}$ is a subfield of K.

Proof. It is clear that $0, 1 \in \overline{L}$. We leave it as an exercise for the reader to prove, on the basis of 11.2, that $\overline{L}$ is a subring of K. We will show that if $x \in \overline{L}$ and $x \neq 0$, then $x^{-1} \in \overline{L}$. So, suppose $x \in \overline{L}$ and $x \neq 0$. Then, by a slight refinement of the argument of 11.2, one can prove the existence of a net $\{\ell_\tau\}_{\tau<cf(K)}$, of non-zero elements of L, such that $\{\ell_\tau\}_{\tau<cf(K)}$ converges to x in K, and, for $\tau < cf(K)$, $|x - \ell_\tau| < |x/2|$. But then $|\ell_\tau| \geq |x/2|$, and so

$$|x^{-1} - \ell_\tau^{-1}| = |x^{-1}\ell_\tau^{-1}| \cdot |x - \ell_\tau| \leq 2/x^2 \cdot |x - \ell_\tau| .$$

It is now simple to see that $\{\ell_\tau^{-1}\}_{\tau<cf(K)}$ converges to x^{-1}, so $x^{-1} \in \overline{L}$. Thus if $x \neq 0$ and $x \in \overline{L}$, then $x^{-1} \in \overline{L}$. This completes the proof.

Lemma 11.4. Suppose $\{x_{0\lambda}\}_{\lambda<\tau}$, $\{x_{1\lambda}\}_{\lambda<\tau}$ are series in K such that $\{x_{0\lambda} \cdot x_{1\lambda}\}_{\lambda<\tau}$ converges to 0 in the topology $\mathcal{J}$. Then either a subseries of $\{x_{0\lambda}\}_{\lambda<\tau}$ converges to 0, or a subseries of $\{x_{1\lambda}\}_{\lambda<\tau}$ converges to 0.

Proof. Suppose $\{x_{0\lambda} \cdot x_{1\lambda}\}_{\lambda<\tau}$ converges to 0. For $i = 0, 1$ let A_i be the set of ordinals λ with $\lambda < \tau$ such that

$|x_{1\lambda}| \geq |x_{(1-i)\lambda}|$. Without loss of generality A_o is cofinal in τ. For λ in A_o, $|x_{0\lambda} \cdot x_{1\lambda}| \geq |x_{1\lambda}|^2$. It follows that $(x_{1\lambda}^2)_{\lambda\epsilon A_o}$ converges to 0, and then it follows that $(x_{1\lambda})_{\lambda\epsilon A_o}$ converges to 0. This proves the result.

<u>Corollary 11.5</u>. Let n be a positive integer, and let $(x_{m\lambda})_{\lambda<\tau}$, $1 \leq m \leq n$, be series in K such that $(\prod_m x_{m\lambda})_{\lambda<\tau}$ converges to 0 in the topology $\mathcal{J}$. Then, for some m_o, a subseries of $(x_{m_o\lambda})_{\lambda<\tau}$ converges to 0.

<u>Proof</u>. Obvious, from 11.4.

Since K is formally real, $i \notin K$ where $i^2 = -1$. Let $Nm\colon K(i) \to K$ be the norm map.

<u>Theorem 11.6</u>. If K and L are real closed, then $\bar{L}$ is real closed.

<u>Proof</u>. Suppose K and L are real closed. Let $\tau = cf(K)$. We will show that $\bar{L}$ is relatively algebraically closed in K.

Suppose $c_1, \ldots, c_n$ are in $\bar{L}$, and α is in K and such that $\alpha^n + c_1\alpha^{n-1} + \cdots + c_n = 0$. By Lemma 11.2 there are nets $(c_{m\lambda})_{\lambda<\tau}$ $(1 \leq m \leq n)$ in L such that $(c_{m\lambda})_{\lambda<\tau}$ converges to c_m. Then $(\alpha^n + c_{1\lambda}\alpha^{n-1} + \cdots + c_{n\lambda})_{\lambda<\tau}$ converges to $\alpha^n + c_1\alpha^{n-1} + \cdots + c_n$, i.e., to 0. By Theorem 1.1, $L(i)$ is algebraically closed, so the equations

$$x^n + c_{1\lambda}x^{n-1} + \cdots + c_{n\lambda} = 0 \qquad (\lambda < \tau)$$

split in $L(i)$. For each $\lambda < \tau$ number the roots of the above equation in $L(i)$ as $\alpha_{1\lambda}, \ldots, \alpha_{n\lambda}$. Then for each λ,

$$\alpha^n + c_{1\lambda}\alpha^{n-1} + \cdots + c_{n\lambda} = \prod_m (\alpha-\alpha_{m\lambda}) .$$

We take norms

$$\mathrm{Nm}(\alpha^n+c_{1\lambda}\alpha^{n-1}+\cdots+c_{n\lambda}) = \prod_m \mathrm{Nm}(\alpha-\alpha_{m\lambda}) .$$

Now

$$\alpha^n + c_{1\lambda}\alpha^{n-1} + \cdots + c_{n\lambda} \in K ,$$

so that

$$\mathrm{Nm}(\alpha^n+c_{1\lambda}\alpha^{n-1}+\cdots+c_{n\lambda}) = (\alpha^n+c_{1\lambda}\alpha^{n-1}+\cdots+c_{n\lambda})^2 .$$

It follows that

$$\{\mathrm{Nm}(\alpha^n+c_{1\lambda}\alpha^{n-1}+\cdots+c_{n\lambda})\}_{\lambda<\tau}$$

converges to 0. Then $\{\prod_m \mathrm{Nm}(\alpha-\alpha_{m\lambda})\}_{\lambda<\tau}$ converges to 0. By the corollary to Lemma 11.4 there follows the existence of a series $\{\alpha_\mu\}_{\mu<\delta}$ in $L(i)$ such that $\{\mathrm{Nm}(\alpha-\alpha_\mu)\}_{\mu<\delta}$ converges to 0. Now clearly there are, for each $\mu < \delta$, ℓ_μ and m_μ in L such that $\alpha_\mu = \ell_\mu + i\,m_\mu$, and then

$$\mathrm{Nm}(\alpha-\alpha_\mu) = \mathrm{Nm}(\alpha-\ell_\mu-i\,m_\mu) = (\alpha-\ell_\mu)^2 + m_\mu^2 .$$

Since $\{\mathrm{Nm}(\alpha-\alpha_\mu)\}_{\mu<\delta}$ converges to 0, it follows that $\{(\alpha-\ell_\mu)^2\}_{\mu<\delta}$ converges to 0, whence $\{\alpha-\ell_\mu\}_{\mu<\delta}$ converges to 0, whence $\{\ell_\mu\}_{\mu<\delta}$ converges to α. Since the ℓ_μ are in L, it follows that $\alpha \in \bar{L}$.

Thus $\bar{L}$ is relatively algebraically closed in K, and since K is real closed it follows that $\bar{L}$ is real closed. This concludes the proof.

11.7. There are certain other notions of closure which it is appropriate to consider. These are incorporated in the following definitions, where (K,L) is any pair of ordered fields.

11.7.1. C_L is defined as the set of those x in K such that for all $\varepsilon > 0$ in L there is an ℓ in L with $|x - \ell| < \varepsilon$.

11.7.2. C_{right} is defined as the set of those x in K such that for all $\varepsilon > 0$ in K there is an ℓ in L with $x < \ell < x + \varepsilon$.

11.7.3. C_{left} is defined as the set of those x in K such that for all $\varepsilon > 0$ in K there is an ℓ in L with $x - \varepsilon < \ell < x$.

11.7.4. $C_{L,right}$ is defined as the set of those x in K such that for all $\varepsilon > 0$ on L there is an ℓ in L with $x < \ell < x + \varepsilon$.

11.7.5. $C_{L,left}$ is defined as the set of those x in K such that for all $\varepsilon > 0$ in L there is an ℓ in L with $x - \varepsilon < \ell < x$.

Lemma 11.8. If L is cofinal in K, then

$$\bar{L} = C_L = C_{right} = C_{left} = C_{L,right} = C_{L,left} .$$

Proof. Suppose L is cofinal in K. Then L is coinitial in K. Then clearly $\bar{L} = C_L$, $C_{right} = C_{L,right}$, and $C_{left} = C_{L,left}$.

We now show that $C_{right} = C_{left}$, and it will clearly follow that $C_{right} = C_{left} = \overline{L}$. Firstly, one sees easily that C_{right} is closed under +. Secondly, since L is cofinal in K, it is clear that $L \subseteq C_{right}$. Now suppose $x \in C_{right}$, and $\varepsilon > 0$. Then there is an $\varepsilon_1 > 0$ in L such that $0 < \varepsilon_1 < \varepsilon$. Now $-\varepsilon_1$ is in C_{right}, since it is in L. It follows that $x - \varepsilon_1$ is in C_{right}. Thus there is an ℓ in L with

$$x - \varepsilon_1 < \ell < (x-\varepsilon_1) + \varepsilon_1 = x .$$

Then $x - \varepsilon < \ell < x$. Since ε was an arbitrary positive element of K, we have thus shown that $C_{right} \subseteq C_{left}$. A similar argument gives the converse, so $C_{right} = C_{left}$. It follows that $\overline{L} = C_{right} = C_{left}$. The lemma follows.

Lemma 11.9. If L is not cofinal in K, then $C_{left} = C_{right} = L = \overline{L}$.

Proof. This is straightforward.

11.10. We now show that if (K,L) is a pair of ordered fields, and L is not cofinal in K, then $C_L = C_{L,right} = C_{L,left}$, but none of these sets is a field.

As usual, let V^L be the ring of L-bounded elements of K, and let I^L be the maximal ideal of non-units of V^L. Then V^L/I^L is an ordered field M, in which L may be canonically embedded. We have the canonical place $\pi: V^L \to M$, and π is weakly order preserving. Let $\overline{L}^{(\pi)}$ be the closure (cf. 11.1) of L in M.

Lemma 11.11. Suppose (K,L) is a pair of ordered fields, where L is not cofinal in K. Then

(i) $C_L \subseteq V^L$;

(ii) $x \in C_L$ if and only if $\Pi(x) \in \overline{L}^{(\pi)}$;

(iii) $C_L = C_{L,\text{right}} = C_{L,\text{left}}$

(iv) C_L is not a field.

Proof. Let (K,L) be a pair with L not cofinal in K.

(i) Suppose $x \in C_L$. Then there is an ℓ in L with $|x - \ell| < 1$. But then $|x| < |\ell| + 1$, so $x \in V^L$. Thus $C_L \subseteq V^L$.

(ii) Suppose first that $x \in C_L$. Let m be an arbitrary positive element of V^L/I^L. Since L is cofinal in V^L/I^L, there is an ℓ in L with $0 < \ell < m$. Now there is an ℓ_1 in L with $|x - \ell_1| < \ell$. But then by the fact that π is weakly-order-preserving, $|\pi(x) - \ell_1| \leq \ell < m$. It follows that if $x \in C_L$, then $\pi(x) \in \overline{L}^{(\pi)}$.

Suppose conversely that $x \in V^L$ and $\pi(x) \in \overline{L}^{(\pi)}$. This means that if m is an arbitrary positive element of V^L/I^L, then there is an ℓ in L such that $|\pi(x) - \ell| < m$. Taking m in L, we conclude that if ℓ_1 is an arbitrary positive element of L there is an ℓ_2 in L such that $|\pi(x) - \ell_2| < \ell_1/2$. But then $|x - \ell_2| < \ell_1$. It follows easily that $x \in C_L$. Thus if $x \in V^L$ and $\pi(x) \in \overline{L}^{(\pi)}$, then $x \in C_L$.

(iii) We leave this as an exercise. One uses the fact that L is cofinal in V^L/I^L, and the same sort of argument as in 11.8.

(iv) Let t be a positive element of I^L. Then $t \in C_L$, by (ii). But clearly $t^{-1} \notin C_L$, since $t^{-1} \notin V^L$.

Section 12. Algebraic Dependence

We refer to [23] for definitions of the basic notions pertaining to algebraic dependence, and for proofs of the basic results. In this section we emphasize certain facts about transcendence bases, needed for Chapter 2.

Lemma 12.1. Let F_1, F_2, F_3 be fields, with $F_1 \subseteq F_3$, and let B_{12} be a transcendence base for F_2 over F_1, and B_{23} a transcendence base for F_3 over F_2. Then $B_{12} \cup B_{23}$ is a transcendence base for F_3 over F_1.

Proof. Let $F_1, F_2, F_3, B_{12}, B_{23}$ be as in the statement of the lemma. F_3 is algebraic over $F_2(B_{23})$, and F_2 is algebraic over $F_1(B_{12})$. Thus F_3 is algebraic over $F_1(B_{12} \cup B_{23})$.

Suppose $B_{12} \cup B_{23}$ is not a transcendence base for F_3 over F_1. Let n be the minimum cardinality of a finite subset of $B_{12} \cup B_{23}$ algebraically dependent over F_1. Select $\xi_1, \ldots, \xi_n$ in $B_{12} \cup B_{23}$ such that $\{\xi_1, \ldots, \xi_n\}$ is algebraically dependent over F_1. We distinguish two cases.

Case 1. $\{\xi_1, \ldots, \xi_n\} \subseteq B_{12}$. This cannot occur since B_{12} is independent over F_1.

Case 2. Some ξ_i is in B_{23}. Without loss of generality, $\xi_o \in B_{23}$. Since no proper subset of $\{\xi_1, \ldots, \xi_n\}$ is algebraically dependent over F_1, since n was chosen minimal, it follows, using the exchange principle for algebraic dependence, that ξ_o depends algebraically on $\{\xi_1, \ldots, \xi_n\}$ over F_1. But then there is an integer k, and $P_o, P_1, \ldots, P_k$ in $F_1[x_1, \ldots, x_n]$ such that not all P_j are 0, and

$\sum_{j=0}^{k} P_j(\xi_1,\ldots,\xi_n) \cdot \xi_0^k = 0$. By the minimality property of $\{\xi_0,\ldots,\xi_n\}$, not all $P_j(\xi_1,\ldots,\xi_n)$ are 0. Without loss of generality, either $\{\xi_1,\ldots,\xi_n\} \cap B_{23} = \emptyset$, or $\{\xi_1,\ldots,\xi_n\} \cap B_{23} = \{\xi_1,\ldots,\xi_{n_1}\}$ for some n_1. In the first case, ξ_0 is algebraic over $F_1(\xi_1,\ldots,\xi_n)$, a subfield of F_2. In the second case, ξ_0 is algebraic over $F_2(\xi_1,\ldots,\xi_{n_1})$. In either case, we contradict the fact that B_{23} is a transcendence base for F_3 over F_2. This concludes the proof.

Lemma 12.2. Let F_1, F_2, F_3 be fields with $F_1 \subseteq F_2 \subseteq F_3$, and let B_{12}, B_{23} be transcendence bases of F_2 over F_1, F_3 over F_2 respectively. Let S_{12}, S_{23} be subsets of B_{12}, B_{23} respectively, and suppose α is in F_2 and algebraic over $F_1(S_{12},S_{23})$. Then α is algebraic over $F_1(S_{12})$.

Proof. Let F_1, F_2, F_3, B_{12}, B_{23}, S_{12}, S_{23}, α be as in the statement of the lemma. Let M be a subset of $S_{12} \cup S_{23}$ minimal with respect to the property that α is algebraic over $F_1(M)$. Let M_1 be $S_{12} \cap M$, and let M_2 be $S_{23} \cap M$. We claim M_2 is empty. Suppose not, and select a fixed ξ in M_2. Then by the minimality of M, and the exchange principle, ξ is algebraic over $F_1(\alpha, M-\{\xi\})$. But then ξ is algebraic over $F_2(M_2-\{\xi\})$, which contradicts the fact that B_{23} is a transcendence base for F_3 over F_2. Thus $M_2 = \emptyset$, and α is algebraic over $F_1(M_1)$. This proves the result.

Lemma 12.3. Suppose (F_2, F_1) is a pair of ordered fields, with card $(F_2) = \text{tr.d. } F_2|F_1$. Then there is a transcendence base B_{12} of F_2 over F_1, such that B_{12} is dense in F_2.

Proof. Suppose (F_2, F_1) is a pair of ordered fields, with $\mathrm{card}(F_2) = \mathrm{tr.d.}\ F_2|F_1 = \aleph_\alpha$. Well order the set of non-empty subintervals of F_2 as $\{I_\lambda\}_{\lambda<\omega_\alpha}$. Let μ be an ordinal less than ω_α, and suppose the set $\{t_\lambda\}_{\lambda<\mu}$ is algebraically independent over F_1, and for each $\lambda < \mu$ $t_\lambda \in I_\lambda$. Let $F_{1\mu}$ be the algebraic closure in F_2 of $F_1(\{t_\lambda\})_{\lambda<\mu}$. Then $\mathrm{tr.d.}\ F_{1\mu}|F_1 < \aleph_\alpha$, so $F_{1\mu} \neq F_2$, and so, by Lemma 3.8, $F_{1\mu}$ does not cover I_μ. Select t_μ in I_μ with $t_\mu \notin F_{1\mu}$. Then t_μ is not algebraically dependent over F_1 on $\{t_\lambda\}_{\lambda<\mu}$, and since $\{t_\lambda\}_{\lambda<\mu}$ is algebraically independent over F_1 it follows that $\{t_\lambda\}_{\lambda\leq\mu}$ is algebraically independent over F_1.

Now by transfinite induction we get a set $\{t_\lambda\}_{\lambda<\omega_\alpha}$ which is algebraically independent over F_1, and such that $t_\lambda \in I_\lambda$ for each $\lambda < \omega_\alpha$. The lemma follows directly.

CHAPTER 2.

ISOMORPHISM THEOREMS

Section 13. η_α-Systems

Let $\mathcal{S}$ be a relational system, with domain A, such that $\mathcal{S}$ has among its relations a unique linear order $<$ on A.

Definition 13.1. Let X and Y be subsets of A. Then $X < Y$ if and only if for every x in X and y in Y we have $x < y$.

Definition 13.2. Let α be an ordinal. Then $\mathcal{S}$ is η_α if and only if for all X and Y of cardinality less than $\aleph_\alpha$, if $X < Y$ there is a t in A such that $X < \{t\} < Y$.

If $\mathcal{S}$ is $\langle A,<\rangle$ and $\mathcal{S}$ is η_α we say simply that $\mathcal{S}$ is an η_α set.

We make the simple observation that if $\mathcal{S}$ is η_α then the cardinal of $\mathcal{S}$ is at least $\aleph_\alpha$, and $\mathcal{S}$ has no first or last element in the order $<$.

We now list some isomorphism theorems for η_α systems of cardinality $\aleph_\alpha$.

Theorem 13.3. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be η_α sets of cardinality $\aleph_\alpha$. Then $\mathcal{S}_1$ is isomorphic to $\mathcal{S}_2$. Indeed, if $\mathcal{J}_1$, $\mathcal{J}_2$ are subsets of $\mathcal{S}_1$, $\mathcal{S}_2$ respectively, both of cardinality less than $\aleph_\alpha$, and φ extends to an isomorphism of $\mathcal{J}_1$ onto $\mathcal{J}_2$, then φ extends to an isomorphism of $\mathcal{S}_1$ onto $\mathcal{S}_2$.

Theorem 13.4. Suppose $\alpha > 0$. Let $\mathcal{S}_1, \mathcal{S}_2$ be η_α D-groups of cardinality $\aleph_\alpha$. Then $\mathcal{S}_1$ is isomorphic to $\mathcal{S}_2$. Indeed, if $\mathcal{J}_1$ and $\mathcal{J}_2$ are D-subgroups of $\mathcal{S}_1, \mathcal{S}_2$ respectively, both of cardinality less than $\aleph_\alpha$, and φ is an isomorphism of $\mathcal{J}_1$ onto $\mathcal{J}_2$ then φ extends to an isomorphism of $\mathcal{S}_1$ onto $\mathcal{S}_2$.

Theorem 13.5. Suppose $\alpha > 0$. Let $\mathcal{S}_1, \mathcal{S}_2$ be η_α real-closed fields of cardinality $\aleph_\alpha$. Then $\mathcal{S}_1$ is isomorphic to $\mathcal{S}_2$. Indeed, if $\mathcal{J}_1, \mathcal{J}_2$ are real-closed subfields of $\mathcal{S}_1, \mathcal{S}_2$ respectively, both of cardinality less than $\aleph_\alpha$, and φ is an isomorphism of $\mathcal{J}_1$ onto $\mathcal{J}_2$, then φ extends to an isomorphism of $\mathcal{S}_1$ onto $\mathcal{S}_2$.

It should be observed that the restriction in 13.4 and 13.5, namely that $\alpha > 0$, is essential. An η_0 system is simply a system whose ordering is a dense ordering without first or last element. It follows that all D-groups and ordered fields are η_0 systems. But, as has been observed many times, not all countable D-groups are isomorphic and not all countable real-closed fields are isomorphic.

Section 14. Pseudo-Completeness

Definition 14.1. Let $\langle K, v, G\rangle$ be a valued field, and λ a limit ordinal. A well-ordered series $\{a_\rho\}_{\rho<\lambda}$ of distinct elements of K is λ-pseudo-Cauchy in $\langle K, v, G\rangle$ if and only if for all $\rho < \sigma < \tau$ $v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$.

Lemma 14.2. Suppose $\{a_\rho\}_{\rho<\lambda}$ is λ-pseudo-Cauchy in $\langle K, v, G\rangle$. Then if $\rho < \sigma$

$$v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho) .$$

Proof. Suppose $\{a_\rho\}_{\rho<\lambda}$ is λ-pseudo-Cauchy. If $\sigma = \rho + 1$, then trivially

$$v(a_{\rho+1} - a_\rho) = v(a_\sigma - a_\rho) .$$

If $\sigma > \rho + 1$ then $v(a_{\rho+1} - a_\rho) < v(a_\sigma - a_{\rho+1})$, so

$$v(a_\sigma - a_\rho) = v(a_\sigma - a_{\rho+1} + a_{\rho+1} - a_\rho) = v(a_{\rho+1} - a_\rho) .$$

Definition 14.3. Suppose $\{a_\rho\}_{\rho>\lambda}$ is λ-pseudo-Cauchy in $\langle K, v, G\rangle$ and $x \in K$. Then x is a limit of $\{a_\rho\}_{\rho<\lambda}$ if and only if, for all $\rho < \lambda$,

$$v(x - a_\rho) = v(a_{\rho+1} - a_\rho) .$$

Definition 14.4. Let α be an ordinal and $\langle K, v, G\rangle$ be a valued field. Then $\langle K, v, G\rangle$ is ω_α-pseudo-complete if and only if, for every $\lambda < \omega_\alpha$, every λ-pseudo-Cauchy series in $\langle K, v, G\rangle$ has a limit in K.

Lemma 14.5. Suppose $\langle K, v, \Gamma\rangle$ is a valued field.

a) If $\{a_\rho\}_{\rho<\lambda}$ is λ-pseudo-Cauchy in $\langle K, v, \Gamma\rangle$, and $\{a_{\rho_\kappa}\}_{\kappa<\mu}$ is a subseries of $\{a_\rho\}_{\rho<\lambda}$, where μ is a limit ordinal, then $\{a_{\rho_\kappa}\}_{\kappa<\mu}$ is μ-pseudo-Cauchy.

b) Suppose $\{a_\rho\}_{\rho<\lambda}$ is λ-pseudo-Cauchy, and $\{a_{\rho_\kappa}\}_{\kappa<\mu}$ is a cofinal subseries of $\{a_\rho\}_{\rho<\lambda}$. Then if x is a limit of $\{a_{\rho_\kappa}\}_{\kappa<\mu}$, x is a limit of $\{a_\rho\}_{\rho<\lambda}$.

Proof.

a) This is obvious from 14.1.

b) Assume the hypothesis of (b). By (a), $\{a_{\rho_\kappa}\}_{\kappa<\mu}$ is μ-pseudo-Cauchy. Suppose x is a limit of $\{a_{\rho_\kappa}\}_{\kappa<\mu}$. Suppose $\rho < \sigma < \lambda$. Select ρ_κ with $\rho_\kappa > \sigma$. Then

$$\begin{aligned} v(x-a_{\rho_\kappa}) &= v(a_{\rho_{\kappa+1}}-a_{\rho_\kappa}) \\ &> v(a_{\rho_\kappa}-a_\rho) \\ &= v(a_\sigma-a_\rho) . \end{aligned}$$

But then

$$\begin{aligned} v(x-a_\rho) &= v(x-a_{\rho_\kappa}+a_{\rho_\kappa}-a_\rho) \\ &= v(a_{\rho_\kappa}-a_\rho) \\ &= v(a_\sigma-a_\rho) . \end{aligned}$$

Therefore x is a limit of $\{a_\rho\}_{\rho<\lambda}$.

Lemma 14.6. Let K be an ordered field, with order $<_K$, such that K is η_α. Let G be an ordered group, with order $<$, and let v be a convex valuation of K onto G. Then $\langle K, v, G\rangle$ is ω_α-pseudo-complete.

Proof. Let K, $<_K$, G, $<$, v be as in the statement of the lemma. Suppose $\{a_\rho\}_{\rho<\lambda}$ is λ-pseudo-Cauchy in $\langle K, v, G\rangle$, where $\lambda < \omega_\alpha$. We are going to show that $\{a_\rho\}_{\rho<\lambda}$ has a limit x in K. We distinguish two cases.

Case 1. There is a cofinal subseries $\{a_{\rho_K}\}_{K<\mu}$ which is monotone in the order $<_K$.

We will show that $\{a_{\rho_K}\}_{K<\mu}$ has a limit x, and then, by 14.5(b), x will be a limit of $\{a_\rho\}_{\rho<\lambda}$. Without loss of generality, therefore, we can assume that $\{a_\rho\}_{\rho<\lambda}$ is monotone in the order $<_K$. Also, since x is a limit of $\{a_\rho\}_{\rho<\lambda}$ if and only if $-x$ is a limit of $\{-a_\rho\}_{\rho<\lambda}$, we can assume without loss of generality that $\{a_\rho\}_{\rho<\lambda}$ is monotone increasing in $<_K$.

For each $\rho < \lambda$ we define γ_ρ as $v(a_{\rho+1}-a_\rho)$. Then, by 14.2, if $\sigma > \rho$, $v(a_\sigma-a_\rho) = \gamma_\rho$. Also, $\{\gamma_\rho\}_{\rho<\lambda}$ is monotone increasing. For each $\rho < \lambda$ we define $b_{\rho+1}$ as $2a_{\rho+1} - a_\rho$.

Suppose $\sigma, \rho < \lambda$.

If $\sigma \leq \rho + 1$,

$$a_\sigma \leq_K a_{\rho+1} <_K a_{\rho+1} + (a_{\rho+1}-a_\rho) = b_{\rho+1} .$$

Suppose $\sigma > \rho + 1$. Then $v(a_\sigma-a_{\rho+1}) = \gamma_{\rho+1}$, and

$$v(b_{\rho+1}-a_{\rho+1}) = v(a_{\rho+1}-a_\rho) = \gamma_\rho < \gamma_{\rho+1} .$$

By convexity, $|b_{\rho+1} - a_{\rho+1}| >_K |a_\sigma - a_{\rho+1}|$. But $a_{\rho+1} <_K a_\sigma$ and $a_{\rho+1} <_K b_{\rho+1}$. It follows that $b_{\rho+1} >_K a_\sigma$.

It follows that for all $\sigma, \rho < \lambda$ we have $a_\sigma <_K b_{\rho+1}$. Since K is η_α there is an x in K such that for all $\sigma, \rho < \lambda$ we have $a_\sigma <_K x <_K b_{\rho+1}$. We claim x is a limit of $\{a_\rho\}_{\rho<\lambda}$. It is clear that if $\rho < \sigma$ $|x - a_\rho| \geq_K |a_\sigma - a_\rho|$, so by convexity $v(x-a_\rho) \leq v(a_\sigma - a_\rho) = \gamma_\rho$. On the other hand, suppose $\rho < \sigma < \tau$. We have $|x - a_\rho| \leq_K |b_{\tau+1} - a_\rho|$, so that

$$v(x-a_\rho) \geq v(b_{\tau+1}-a_\rho) = v(a_{\tau+1}-a_\tau+a_{\tau+1}-a_\rho) .$$

Now $v(a_{\tau+1}-a_\tau) = \gamma_\tau$, and $v(a_{\tau+1}-a_\rho) = \gamma_\rho$. Therefore

$$v(a_{\tau+1}-a_\tau+a_{\tau+1}-a_\rho) = \gamma_\rho = v(a_\sigma-a_\rho) .$$

Therefore $v(x-a_\rho) \geq v(a_\sigma-a_\rho)$. Combining our results, we have proved that $v(x-a_\rho) = v(a_\sigma-a_\rho)$, if $\rho < \sigma$, so that x is a limit of $\{a_\rho\}_{\rho<\lambda}$.

Case 2. No cofinal subseries of $\{a_\rho\}_{\rho<\lambda}$ is monotone in the order $<_K$.

We observe first that if $a_\rho <_K a_{\rho+1}$ and $\sigma > \rho + 1$, then $a_\rho <_K a_\sigma$. For, since $\{a_\rho\}_{\rho<\lambda}$ is pseudo-Cauchy, $v(a_\sigma-a_{\rho+1}) > v(a_{\rho+1}-a_\rho)$, and so by convexity $|a_\sigma - a_{\rho+1}| <_K |a_{\rho+1} - a_\rho|$. The result follows. By a similar argument we can easily show that if $a_{\rho+1} <_K a_\rho$ and $\sigma > \rho + 1$, then $a_\sigma <_K a_\rho$. A simple consequence of these results is that if σ is a limit ordinal with $\sigma < \lambda$, and τ is an ordinal less than σ so that $\{a_\rho\}_{\tau\leq\rho<\sigma}$ is monotone, then $\{a_\rho\}_{\tau\leq\rho\leq\sigma}$ is monotone.

Let S_1 be $\{\rho | a_\rho <_K a_{\rho+1}\}$, and let S_2 be $\{\rho | a_{\rho+1} <_K a_\rho\}$. Then by the remarks above, and since we are in Case 2, S_1 and S_2 are both cofinal in λ. Since the a_ρ are distinct, $S_1 \cup S_2 = \lambda$.

Let Σ_1 be $\{a_\rho | \rho \in S_1\}$, and let Σ_2 be $\{a_\rho | \rho \in S_2\}$. Let a_{ρ_1} be a member of Σ_1, and a_{ρ_2} a member of Σ_2. It is clear from the results two paragraphs back, that $a_{\rho_1} <_K a_{\rho_2}$.

Since K is η_α, there is an x such that, for all ρ_1 in S_1 and ρ_2 in S_2, $a_{\rho_1} <_K x <_K a_{\rho_2}$. But now, by an argument just like that in Case 1, we may show that x is a limit of $\{a_\rho\}_{\rho<\lambda}$.

Thus in all cases $\{a_\rho\}_{\rho<\lambda}$ has a limit in K, and the result is proved.

Section 15. Extension Theorems

Theorem 15.1. Suppose $\alpha > 0$. Suppose K_1 and K_2 are real-closed fields. Suppose M_1 and M_2 are real-closed subfields of K_1, K_2 respectively, such that for $i = 1, 2$, M_i is M_i-maximal in K_i, and tr.d. $K_i | M_i = \aleph_\alpha$. Suppose H_1 and H_2 are real-closed subfields of K_1, K_2 respectively, such that, for $i = 1, 2$, $M_i \subset H_i$, and tr.d. $H_i | M_i < \aleph_\alpha$. Suppose $\langle \varphi, \psi \rangle$ is an analytic isomorphism of $\langle H_2, v_{K_2}^{M_2} \restriction H_1, v_{K_1}^{M_1}[H_1^*] \rangle$ onto $\langle H_2, v_{K_2}^{M_2} \restriction H_2, v_{K_2}^{M_2}[H_2^*] \rangle$, such that $\varphi[M_1] = M_2$. Suppose ψ extends to an order-isomorphism ψ of $v_{K_1}^{M_1}[K_1^*]$ onto $v_{K_2}^{M_2}[K_2^*]$. Then φ extends to an isomorphism φ' of K_1 onto K_2, such that $\langle \varphi', \psi' \rangle$ is an analytic isomorphism.

We are going to deduce Theorem 15.1 from a very important theorem of Ax and Kochen. We need a few algebraic preliminaries.

Definition 15.2. Let $\langle K, v, G \rangle$ be a valued field. Then $\langle K, v, G \rangle$ has the uniqueness property if and only if v has a unique extension to every finite algebraic extension of K.

Lemma 15.3. Suppose K is real closed, and v is a convex valuation of K onto G. Then $\langle K, v, G \rangle$ has the uniqueness property.

Proof. Suppose K is real closed and v is a convex valuation of K onto G. By 1.3, K has only one proper algebraic extension, namely $K(i)$.

It is a standard fact of valuation theory that there exists an extension of v to $K(i)$.

Suppose v_1 is a valuation of $K(i)$ onto G_1, with v_1 extending v. Let V, V_1 be the valuation rings of v, v_1 respectively, and π,

π_1 the places of v, v_1 respectively. Now

$$2v_1(i) = v_1(i^2) = v_1(-1) = v(-1) = 0 .$$

Since G_1 is torsion-free, $v_1(i) = 0$. Also,

$$(\pi_1(i))^2 = \pi_1(i^2) = \pi_1(-1) = -1 .$$

Now, since K is real closed and v is convex, $\pi[V]$ is real closed, by an argument like that in 6.8. Let j be $\pi_1(i)$. Then $j^2 = -1$, and $\pi_1[V_1]$ is the algebraic closure of $\pi[V]$.

Let Z be an arbitrary element of $K(i)$. Z has a unique representation as $x + iy$, where $x, y \in K$. If $x/y \in V$,

$$\begin{aligned} v_1(Z) &= v_1(y) + v_1(x/y + i) \\ &= v(y) + v_1(i) + v_1(1+i(-x/y)) \\ &= v(y) + v_1(1+i(-x/y)) . \end{aligned}$$

If $y/x \in V$, $v_1(Z) = v_1(x) + v_1(1+i \cdot y/x)$. Thus we see that v_1 is determined by the values $v_1(1+it)$ where $t \in V$. But if $t \in V$, $1 + it \in V_1$ and

$$\pi_1(1+it) = 1 + \pi_1(it) = 1 + j\pi(t) \neq 0 .$$

Then $v_1(1+it) = 0$.

Thus v_1 is unique, and the lemma is proved.

<u>15.4</u>. As formulated in 15.2, the uniqueness property is defined by quantification over field extensions. It is a remarkable and important fact that this second-order quantification is not essential for the definition of the uniqueness property.

It has been shown that an arbitrary valued field $\langle K, v, G\rangle$ has the uniqueness property if and only if Hensel's Lemma holds for $\langle K, v, G\rangle$. (In the notes we give references for this result.) In accordance with this result, we shall say that $\langle K, v, G\rangle$ is Henselian if and only if $\langle K, v, G\rangle$ has the uniqueness property.

We do not use the following result 15.5, but state it here, without proof, on the grounds of its relevance and importance.

Lemma 15.5. Suppose $\langle K, v, G\rangle$ is Henselian, with place π and valuation ring V, and suppose $\pi[V]$ has characteristic 0. Let M be any subfield of K maximal with respect to the property that $v^*[M] = 0$. Then $\pi: M \to \pi[V]$ is an isomorphism.

Remark. By 15.3, 6.8 is a special case of 15.5.

Definition 15.6. Let $\langle F, v, G\rangle$ be a valued field. A map $\tau: G \to F^*$ is a cross section of v if τ is a homomorphism of G into the multiplicative group F^*, and $v \circ \tau$ is the identity on G.

Lemma 15.7. Suppose F is real closed and v is a convex valuation of F onto G. Then there exists a cross section τ of v.

Proof. Suppose F is real closed, and v is a valuation of F onto G. Then G is a D-group, by 8.6, and so may be construed as a vector space over Q, the rationals. Let $\{g_i\}_{i\in I}$ be a basis for G over Q. For each i in I select a positive x_i in K^* such that $v(x_i) = g_i$. By $x_i^{m/n}$, where $m, n \in Z$, we understand the positive n^{th} root of x_i^m. If $g \in G$, there is an integer k, and $i_1, \ldots, i_k$ in I, such that $g = \sum_{j=1}^{k} r_j \cdot x_{i_j}$, where the r_j are in Q. We define $\tau(g)$ by

$\tau(g) = \prod_{j=1}^{k} x_{i_j}^{r_j}$. It is clear that τ is a cross section of v.

Remark. The above proof clearly shows that if H is a subfield of K then there is a cross section τ such that $\tau[v[H^*]] \subset H$.

Definition 15.8. Let $\langle K, v, G, \tau\rangle$ be a valued-field with cross-section τ. Let π be the place of v. A subfield H of K is normalized, with respect to τ, if for every g in $v[H^*]$ there is an x in H such that $v(x) = g$ and $\pi(x \cdot \tau(-g)) = 1$. (In particular, if $\tau[v[H^*]]$ is a subset of H, H is normalized with respect to τ.)

Definition 15.9. For $i = 1, 2$ let $\langle K_i, v_i, G_i, \tau_i\rangle$ be valued fields with cross section, and let π_i be the place of v_i. Let F_1, F_2 be subfields of K_1, K_2 respectively, and let $\langle \varphi, \psi\rangle$ be an analytic isomorphism of $\langle F_1, v_1 \upharpoonright F_1, v_1[F_1^*]\rangle$ onto $\langle F_2, v_2 \upharpoonright F_2, v_2[F_2^*]\rangle$. Then $\langle \varphi, \psi\rangle$ is a norm-isomorphism if and only if for every g in $v_1[F_1^*]$ there is an x in F_1 such that $\pi_1(x \cdot \tau_1(-g)) = 1$, and $\pi_2(\varphi(x) \cdot \tau_2(-\psi(g))) = 1$.

We can now state the theorem of Ax and Kochen.

Theorem 15.10. Suppose $\alpha > 0$. Suppose $\langle K_1, v_1, G_1\rangle$ and $\langle K_2, v_2, G_2\rangle$ are Henselian valued fields with fixed cross-sections τ_1 and τ_2 respectively. Suppose that the residue-class fields of v_1 and v_2 are of characteristic 0. For $i = 1, 2$ let M_i be a subfield of K_i maximal with respect to the property that $v_i(M_i^*) = 0$. Suppose tr.d. $K_i | M_i = \aleph_\alpha$, for $i = 1, 2$, and $\langle K_i, v_i, G_i\rangle$ is ω_α-pseudo complete. Suppose, for $i = 1, 2$, that H_i is a normalized, relatively algebraically closed subfield of K_i such that $M_i \subset H_i$ and tr.d. $H_i | M_i < \aleph_\alpha$. Suppose

$\langle \varphi, \psi \rangle$ is a norm-isomorphism of H_1 onto H_2, such that $\varphi[M_1] = M_2$. Suppose ψ extends to an order-isomorphism ψ' of G_1 onto G_2. Then φ extends to an isomorphism φ' of K_1 onto K_2, such that $\langle \varphi', \psi' \rangle$ is an analytic isomorphism.

<u>Proof of 15.1</u>. As formulated, the hypotheses of 15.1 closely resemble those of 15.10. To prove 15.1 we need only the observations below. We assume the hypotheses of 15.1.

a) Let G_i be $v_{K_i}^{m_i}[K_i^*]$ for $i = 1, 2$. Then, by 15.3, $\langle K_i, v_{K_i}^{M_i}, G_i \rangle$ is Henselian for $i = 1, 2$.

b) By 14.6, $\langle K_i, v_{K_i}^{M_i}, G_i \rangle$ is ω_α-pseudo complete, since K_i is η_α.

c) By the remarks after 15.7 we select a cross section τ_1 of $v_{1_{K_1}}^{M_1}$ such that $\tau_1[v_{K_1}^{M_1}[H_1]] \subset H_1$. We then define a cross-section τ_2 of $v_{K_2}^{M_2}$ by: $\tau_2 = \varphi \circ \tau_1 \circ \psi^{-1}$. Then H_1 and H_2 are normalized, with respect to τ_1 and τ_2, and $\langle \varphi, \psi \rangle$ is a norm-isomorphism. For $i = 1, 2$, H_i is real closed, and so, by 1.7, H_i is relatively algebraically closed in K_i.

Thus we see that all the hypotheses of 15.10 are satisfied, and 15.11 follows.

<u>15.11</u>. We recall from field theory the notion of linear disjointness. Suppose P and L are subfields of a field K. It is a standard result that the following two conditions are equivalent.

(i) Every finite subset of P that is linearly independent over $L \cap P$ is also linearly independent over L.

(ii) Every finite subset of L that is linearly independent over $L \cap P$ is also linearly independent over P.

When either condition holds we say that L and P are linearly disjoint over $L \cap P$. It is easily seen that if L and P are linearly disjoint over $L \cap P$, and B is a transcendence base for L over $L \cap P$ then B is algebraically independent over P, so that if LP is the compositum of L and P then B is a transcendence base for LP over P.

Now suppose (K,L) is a pair of fields and P is a subfield of K. We say that P is L-independent if P and L are linearly disjoint over $L \cap P$.

15.12. Suppose (K,L) is a pair of ordered fields, and P is a subfield of K. P is said to be L-placed if and only if

$$\pi^L[V^L \cap P] = \pi^L[V^L \cap P \cap L] .$$

(In more picturesque language, every element of P that is L-bounded in K is L-infinitesimally close to an element of $P \cap L$.)

15.13. Suppose m and n are finite ordinals, and σ is an arbitrary ordinal. Then $\sigma \equiv m \pmod n$ if and only if either σ is a positive finite ordinal and $\sigma \equiv m \pmod n$ in the usual sense, or σ is $\lambda + \sigma_1$, where λ is a limit ordinal and σ_1 is a positive finite ordinal and $\sigma_1 \equiv m \pmod n$ in the usual sense.

15.14. If K is a real-closed field and P is a subfield of K, $\operatorname{Re}P$ is to be the real-closure of P in K, i.e., the relative algebraic closure of P in K.

Theorem 15.15. Suppose $\alpha > 0$. For $i = 1, 2$, suppose K_i is a real-closed field, and M_i is an η_α real-closed subfield of K_i, of cardinality $\aleph_\alpha$, such that M_i is M_i-maximal in K_i. For $i = 1, 2$, suppose P_i is an M_i-placed, M_i-independent, real-closed subfield of K_i, of cardinality less than $\aleph_\alpha$. For $i = 1, 2$, let T_i be the real closure of M_iP_i in K_i. Suppose φ is an isomorphism of $(P_1, P_1 \cap M_1)$ onto $(P_2, P_2 \cap M_2)$. Then φ extends to an isomorphism φ' of (T_1, M_1) onto (T_2, M_2).

Proof. Assume the hypotheses of the theorem. It is clear for $i = 1, 2$ that $\text{tr.d. } M_i | M_i \cap P_i = \aleph_\alpha$. Using 12.3, we select transcendence bases B_1, B_2 for M_1 over $M_1 \cap P_1$, M_2 over $M_2 \cap P_2$ respectively, such that B_1 is dense in M_1 and B_2 is dense in M_2. Well-order B_i, for $i = 1, 2$, as $\{b_i^\lambda\}_{\lambda<\omega_\alpha}$. As observed earlier, B_i is a transcendence base for T_i over P_i, for $i = 1, 2$.

We are going to define inductively for each $\lambda < \omega_\alpha$, real-closed fields P_i^λ, with $P_i^\lambda \subset T_i$, and isomorphisms $\varphi^\lambda: P_1^\lambda \xrightarrow{\text{onto}} P_2^\lambda$ such that:

i) $P_i^0 = P_i$ and $\varphi_\lambda^0 = \varphi$;

ii) If $\rho < \lambda$, $P_i^\rho \subset P_i^\lambda$ and $\varphi^\rho = \varphi^\lambda \upharpoonright P_1^\rho$;

iii) a) If $\lambda \equiv 0 \pmod 2$ then $P_1^\lambda = \operatorname{Re}P_1^{(\lambda-1)}(b_1^{v_1})$, where

$v_1 = v_1(\lambda) =$ the least ordinal $v < \omega_\alpha$ such that $b_1^v \notin P_1^{(\lambda-1)}$,

and $P_2^\lambda = \operatorname{Re}P_2^{(\lambda-1)}(b_2^{v_2})$, where $b_2^{v_2} \in B_2$, and $\varphi^\lambda(b_1^{v_1}) = b_2^{v_2}$;

b) If $\lambda \equiv 1 \pmod 2$ then $P_1^\lambda = \mathrm{Re}P_1^{(\lambda-1)}(b_1^{\mu_1})$, where $b_1^{\mu_1} \in B_1$, and $P_2^\lambda = \mathrm{Re}P_2^{(\lambda-1)}(b_2^{\mu_2})$, where $\mu_2 = \mu_2(\lambda)$ = the least $\mu < \omega_\alpha$ such that $b_2^\mu \notin P_2^{(\lambda-1)}$, and $\varphi^\lambda(b_1^{\mu_1}) = b_2^{\mu_2}$;

c) If λ is a limit ordinal, $P_i^\lambda = \bigcup_{\rho<\lambda} P_i^\rho$, and φ^λ is the union of the maps φ^ρ, for $\rho < \lambda$.

We show how to achieve the inductive step. The case (iii)(c) is easy, and (iii)(b) is just like (iii)(a), with the roles of B_1 and B_2 reversed. (iii)(a). Suppose $\lambda \equiv 0 \pmod 2$, and we have carried out the construction for all ordinals $\rho < \lambda$.

The following may easily be verified, using transfinite induction:

1. $P_i^{(\lambda-1)} \subset T_i$;
2. $P_i^{(\lambda-1)}$ has cardinal less than $\aleph_\alpha$;
3. $P_i^{(\lambda-1)}$ is M_i-placed;
4. $\varphi^{(\lambda-1)}$ maps $M_1 \cap P_1^{(\lambda-1)}$ onto $M_2 \cap P_2^{(\lambda-1)}$.

(1) and (2) are clear, (3) is obtained by repeated use of 7.4 and 7.5, and (4) is obtained by repeated use of 12.2.

By (1) and (2), and the fact that tr.d. $T_1 | P_1 = \aleph_\alpha$, we see that there are $v < \omega_\alpha$ such that $b_1^v \notin P_1^{(\lambda-1)}$. Let v_1 be the least such v. $b_1^{v_1}$ makes the cut $\mathcal{C}(P_1^{(\lambda-1)}, b_1^{v_1})$ in $P_1^{(\lambda-1)}$. Let Δ be $\varphi^{(\lambda-1)}[\mathcal{C}(P_1^{(\lambda-1)}, b_1^{v_1})]$, and let Γ be $P_2^{(\lambda-1)} - \Delta$. Then $\Delta < \Gamma$. Then $\Delta \cap M_2 < \Gamma \cap M_2$, and by the η_α property of M_2 there is a non-empty interval I of M_2 such that $\Delta \cap M_2 < I < \Gamma \cap M_2$. Since B_2

is dense in M_2, B_2 intersects I. Let v_2 be the least v such that $b_2^v \in I$. We claim $\mathcal{C}(P_2^{(\lambda-1)}, b_2^{v_2}) = \Delta$. Suppose $t \in P_2^{(\lambda-1)}$ and $t < b_2^{v_2}$. Since $b_2^{v_2} \in M_2$, t is either M_2-bounded, or $t < 0 < b_2^{v_2}$.

Case A. t is M_2-bounded. Since $P_2^{(\lambda-1)}$ is M_2-placed, there is a unique m in $M_2 \cap P_2^{(\lambda-1)}$ such that $t - m$ is M_2-infinitesimal. Suppose $\varphi^{(\lambda-1)}(m_1) = m$, and $\varphi^{(\lambda-1)}(t_1) = t$. Then $m_1 \in M_1$, and $t_1 - m_1$ is $M_1 \cap P^{(\lambda-1)}$- infinitesimal. Since $P_1^{(\lambda-1)}$ is M_1-placed, $t_1 - m_1$ is M_1-infinitesimal. Since $m \in M_2 \cap P_2^{(\lambda-1)}$, and $m < b_2^{v_2}$, it follows that $m_1 < b_1^{v_1}$. But since $b_1^{v_1} \in M_1$, and $t_1 - m_1$ is M_1-infinitesimal, it follows that $t_1 < b_1^{v_1}$. But then since $\varphi^{(\lambda-1)}(t_1) = t$, it follows that $t \in \Delta$.

Case B. $t < 0 < b_2^{v_2}$. Since $0 \in M_2$, it follows that $0 < b_1^{v_1}$. Suppose $\varphi^{(\lambda-1)}(t_1) = t$. Then $t_1 < 0 < b_1^{v_1}$. Therefore $t \in \Delta$.

Thus in either case we have shown that if $t \in P_2^{(\lambda-1)}$ and $t < b_2^{v_2}$, then $t \in \Delta$. Conversely, we suppose $u \in \Delta$, and show that $u < b_2^{v_2}$. Since $u \in \Delta$, u is M_2-bounded, and so, since $P_2^{(\lambda-1)}$ is M_2-placed, there is an m in M_2 such that $u - m$ is M_2-infinitesimal. Then, since $b_2^{v_2} \in M_2$,

$$u < b_2^{v_2} \iff m < b_2^{v_2} .$$

Thus we may assume without loss of generality that $u \in M_2$. Suppose $\varphi^{(\lambda-1)}(m_1) = u$. Then $m_1 \in M_1$, and, since $u \in \Delta$, $m_1 < b_1^{v_1}$. It follows that $u = \varphi^{(\lambda-1)}(m_1) < b_2^{v_2}$. This concludes the proof that $(P_2^{(\lambda-1)}, b_2^{v_2}) = \Delta$.

It now follows by familiar principles that $\varphi^{(\lambda-1)}$ extends to an isomorphism $\varphi^{(\lambda)}$ of $\operatorname{Re} P_1^{(\lambda-1)}(b_1^{v_1})$ onto $\operatorname{Re} P_2^{(\lambda-1)}(b_2^{v_2})$ such that $\varphi^{(\lambda)}(b_1^{v_1}) = b_2^{v_2}$. Thus the inductive step is achieved.

It is easily verified, for $i = 1, 2$, that $\bigcup_{\lambda<\omega_\alpha} P_i^\lambda = T_i$, and that φ', the union of the φ^{λ} for $\lambda < \omega_{\alpha}$, is an isomorphism of T_1 onto T_2, mapping M_1 onto M_2. Clearly φ' extends φ, and we are through.

We are going to combine 15.1 and 15.15 to get the final theorem of this section. First we need a lemma.

Lemma 15.16. Suppose K is a real-closed field and M is a subfield. Suppose T is a real-closed subfield of K, with $M \subset T$. Let m be the dimension of $v^M[T]$, construed as a vector space over Q. Then $m \leq$ tr.d. $T|M$.

Proof. Suppose $g_1, \ldots, g_k$ are in $v^M[T]$ and linearly independent over Q. Select $x_1, \ldots, x_k$ in T such that for $1 \leq i \leq k$, $v^M(x_i) = g_i$. Suppose

$$\sum_{k\text{-tuples}} m_{\langle n_1,\ldots,n_k\rangle} x_1^{n_1} \cdots x_k^{n_k} = 0$$

where the summation is taken over all k-tuples of non-negative integers, each $m_{\langle n_1,\ldots,n_k\rangle}$ is in M, only finitely $m_{\langle n_1,\ldots,n_k\rangle}$ are non-zero, and $m_{\langle 0,0,\ldots,0\rangle}$ is non-zero. It follows that there are distinct k-tuples, $\langle n_1,\ldots,n_k\rangle$ and $\langle n_1',\ldots,n_k'\rangle$, such that

$$v^M(m_{\langle n_1,\ldots,n_k\rangle} x_1^{n_1} \cdots x_k^{n_k}) = v^M(m_{\langle n_1',\ldots,n_k'\rangle} x_1^{n_1'} \cdots x_k^{n_k'}) .$$

Now $v^M(m_{\langle n_1,\ldots,n_k\rangle}) = v^M(m_{\langle n_1',\ldots,n_k'\rangle}) = 0$. Therefore

$$\sum_{i=1}^{k} v^M(x_i^{n_i}) = \sum_{i=1}^{k} v^M(x_i^{n_i'}) .$$

Therefore

$$\sum_{i=1}^{k} n_i g_i = \sum_{i=1}^{k} n_i' \cdot g_i \ .$$

But this gives a dependence between the g's, which is a contradiction. Thus $x_1, \ldots, x_k$ are algebraically independent over M. The result follows.

Theorem 15.17. Suppose $\alpha > 0$. Suppose K_1 and K_2 are η_α real-closed fields. For $i = 1, 2$, suppose M_i is an η_α real-closed subfield of K_i, of cardinality $\aleph_\alpha$, such that tr.d. $K_i | M_i = \aleph_\alpha$, M_i is M_i-maximal in K_i, and $v^{M_i}[K_i]$ is η_α of cardinality $\aleph_\alpha$. For $i = 1, 2$, suppose P_i is an M_i-placed, M_i-independent, real-closed subfield of K_i, of cardinality less than $\aleph_\alpha$. Suppose φ is an isomorphism of $(P_1, P_1 \cap M_1)$ onto $(P_2, P_2 \cap M_2)$. Then φ extends to an isomorphism of (K_1, M_1) onto (K_2, M_2).

Proof. Assume the hypotheses of the theorem. For $i = 1, 2$, let T_i be the real closure of M_iP_i in K_i. Then by 15.15, φ extends to an isomorphism φ' of (T_1, M_1) onto (T_2, M_2). It is clear that φ' induces an isomorphism ψ' of $v^{M_1}[T_1]$ onto $v^{M_2}[T_2]$, so that $\langle \varphi', \psi' \rangle$ is an analytic isomorphism. Now since P_i has cardinality less than $\aleph_\alpha$, it follows that tr.d. $P_i | P_i \cap M_i < \aleph_\alpha$, and so, since P_i is M_i-independent, tr.d. $T_i | M_i < \aleph_\alpha$. By Lemma 15.16, the dimension of $v^{M_i}[T_i]$ is less than $\aleph_\alpha$, and so $v^{M_i}[T_i]$ has cardinal less than $\aleph_\alpha$. But then by 13.4, ψ' extends to an isomorphism ψ'' of $v^{M_1}[K_1]$ onto

$v^{M_2}[K_2]$. Thus Theorem 15.1 applies, and φ' extends to an isomorphism φ'' of (K_1,M_1) onto (K_2,M_2). This proves the theorem.

Corollary 15.18. Suppose $\alpha > 0$. Suppose K_1 and K_2 are η_α real-closed fields. For $i = 1, 2$, suppose M_i is an η_α real-closed subfield of K_i, of cardinality $\aleph_\alpha$, M_i is M_i-maximal in K_i, and $v^{M_i}[K_i]$ is η_α of cardinality $\aleph_\alpha$. Then there is an isomorphism of (K_1,M_1) onto (K_2,M_2).

Proof. Assume the hypotheses of the theorem. Let P_i be (the copy of) the real algebraic numbers in K_i. Then P_i is M_i-independent, M_i-placed, and of cardinality less than $\aleph_\alpha$. Let $\aleph_\alpha$. Let φ be the "identity" isomorphism of P_1 onto P_2. Then by 15.17, φ extends to an isomorphism of (K_1,M_1) onto (K_2,M_2).

Section 16. The Dense Case

Theorem 16.1. Suppose $\alpha > 0$. Suppose, for $i = 1, 2$, that K_i is an η_α real-closed field of cardinality $\aleph_\alpha$, and L_i is a real-closed subfield of K_i such that L_i is dense in K_i and tr.d. $K_i | L_i = \aleph_\alpha$. Suppose P_i, for $i = 1, 2$, is an L_i-independent real-closed subfield of K_i, of cardinality less than $\aleph_\alpha$, such that tr.d. $K_i | L_i P_i = \aleph_\alpha$. Suppose φ is an isomorphism of $(P_1, P_1 \cap L_1)$ onto $(P_2, P_2 \cap L_2)$. Then φ extends to an isomorphism φ' of (K_1, L_1) onto (K_2, L_2).

Proof. Assume the hypotheses of the theorem. We observe that $P_i \cap L_i$ is relatively algebraically closed in K_i, and so by 1.7, $P_i \cap L_i$ is real closed.

It is clear that L_1 and L_2 are η_α of cardinality $\aleph_\alpha$, and tr.d. $L_i | P_i \cap L_i = \aleph_\alpha$, for $i = 1, 2$.

By 12.3, we select a transcendence base B_i of L_i over $L_i \cap P_i$, with B_i dense in L_i. Then B_i is dense in K_i. Well-order B_i as $\{b_i^\lambda\}_{\lambda<\omega_\alpha}$. Again, by 12.3, we select a transcendence base C_i of K_i over $L_i P_i$, with C_i dense in K_i. Well-order C_i as $\{c_i^\lambda\}_{\lambda<\omega_\alpha}$. We observe that $B_i \cup C_i$ is a transcendence base for K_i over P_i.

We will define inductively, for each $\lambda < \omega_\alpha$, real-closed subfields P_i^λ of K_i, and isomorphisms φ^λ: P_1^λ onto P_2^λ, such that:

(i) $P_1^0 = P_1$, $P_2^0 = P_2$, $\varphi^0 = \varphi$;

(ii) If $\rho < \lambda$, then $P_i^\rho \subset P_i^\lambda$, and $\varphi^\rho = \varphi^\lambda \upharpoonright P_1^\rho$:

(iii) (a) If $\lambda \equiv 0 \pmod 4$, then $P_i^\lambda = \mathrm{Re}P_i^{(\lambda-1)}(b_i^{\nu_i})$, where $\nu_1 = \nu_1(\lambda)$ = the smallest ordinal $\nu < \omega_\alpha$ such that $b_1^\nu \notin P_1^{(\lambda-1)}$, and ν_2 is such that $b_2^{\nu_2} \notin P_2^{(\lambda-1)}$, and $\varphi^\lambda(b_1^{\nu_1}) = b_2^{\nu_2}$;

(b) If $\lambda \equiv 1 \pmod 4$, then $P_i^\lambda = \mathrm{Re}P_i^{(\lambda-1)}(b_i^{\mu_i})$, where μ_1 is such that $b_1^{\mu_1} \notin P_1^{(\lambda-1)}$, and $\mu_2 = \mu_2(\lambda)$ = the smallest ordinal $\mu < \omega_\alpha$ such that $b_2^\mu \notin P_2^{(\lambda-1)}$, and $\varphi^\lambda(b_1^{\mu_1}) = b_2^{\mu_2}$;

(c) If $\lambda \equiv 2 \pmod 4$, then $P_i^\lambda = \mathrm{Re}P_i^{(\lambda-1)}(c_i^{\delta_i})$, where $\delta_1 = \delta_1(\lambda)$ = the smallest ordinal $\delta < \omega_\alpha$ such that $c_1^\delta \notin P_1^{(\lambda-1)}$, and δ_2 is such that $c_2^{\delta_2} \notin P_2^{(\lambda-1)}$, and $\varphi^\lambda(c_1^{\delta_1}) = c_2^{\delta_2}$;

(d) If $\lambda \equiv 3 \pmod 4$, then $P_i^\lambda = \mathrm{Re}P_i^{(\lambda-1)}(c_i^{\varepsilon_i})$, where ε_i is such that $c_1^{\varepsilon_1} \notin P_1^{(\lambda-1)}$, and $\varepsilon_2 = \varepsilon_2(\lambda)$ = the smallest ordinal $\varepsilon < \omega_\alpha$ such that $c_2^\varepsilon \notin P_2^{(\lambda-1)}$, and $\varphi^\lambda(c_1^{\varepsilon_1}) = c_2^{\varepsilon_2}$;

(e) If λ is a limit ordinal, then $P_i^\lambda = \bigcup_{\rho<\lambda} P_i^\rho$, and φ^λ is such that $\varphi^\lambda \upharpoonright P_1^\rho = \varphi^\rho$.

We show how to perform the inductive step. The case (iii)(e) is trivial. We discuss (iii)(a) and (iii)(c). The other cases are similar, with the roles of P_1 and P_2 reversed.

Case (iii)(a). Suppose $\lambda \equiv 0 \pmod 4$, and we have accomplished the construction for all ordinals less than λ. It is easily seen that the cardinality of $P_i^{(\lambda-1)}$ is less than $\aleph_\alpha$, for $i = 1, 2$. Thus there exist ordinals $\nu < \omega_\alpha$ such that $b_1^\nu \notin P_1^{(\lambda-1)}$. Let $\nu_1(\lambda)$ be the least such ordinal ν. $b_1^{\nu_1(\lambda)}$ makes the cut $\mathcal{C}(P_1^{(\lambda-1)}, b_1^{\nu_1(\lambda)})$ in $P_1^{(\lambda-1)}$. Consider $\varphi^{(\lambda-1)}[\mathcal{C}(P_1^{(\lambda-1)}, b_1^{\nu_1(\lambda)})]$. This set is of cardinality less than $\aleph_\alpha$. By the η_α property of K_2 it follows that there is a non-empty interval I in K_2 such that for all x in I $\mathcal{C}(P_2^{(\lambda-1)},x) = \varphi^{(\lambda-1)}[\mathcal{C}(P_1^{(\lambda-1)}, b_1^{\nu_1(\lambda)})]$. Since B_2 is dense in K_2, B_2 intersects I, and, of course, $I \cap P_2^{(\lambda-1)} = \emptyset$. Let ν_2 be the least ordinal $\nu < \omega_\alpha$ such that $b_2^\nu \in I$. By 1.11 and 1.8 it follows that $\varphi^{(\lambda-1)}$ has a unique extension to an isomorphism

φ^λ: $\mathrm{Re}P_1^{(\lambda-1)}(b_1^{\nu_1}) \cong \mathrm{Re}P_2^{(\lambda-1)}(b_2^{\nu_2})$, such that $\varphi^\lambda(b_1^{\nu_1}) = b_2^{\nu_2}$.

We set $P_1^\lambda = \mathrm{Re}P_1^{(\lambda-1)}(b_1^{\nu_1})$, and $P_2^\lambda = \mathrm{Re}P_2^{(\lambda-1)}(b_2^{\nu_2})$, and the induction step is achieved.

Case (iii)(c). This is analogous to the previous case, except that we work with the C_i rather than B_i, and use the density of C_2. We omit the details.

Thus the construction can be carried out. But then it is completely clear that $\bigcup_{\lambda<\omega_\alpha} P_i^\lambda = K_i$, and that the union of the φ^λ, for $\lambda < \omega_\alpha$, is an isomorphism of K_1 onto K_2. Let φ' be this union map. Then φ' maps B_1 onto B_2, and so φ' maps L_1 onto L_2. Obviously φ' extends φ. The proof is now complete.

Corollary 16.2. Suppose $\alpha > 0$. Suppose, for $i = 1, 2$, that (K_i, L_i) are pairs of η_α real-closed fields of cardinality $\aleph_\alpha$, such that L_i is dense in K_i and tr.d. $K_i | L_i = \aleph_\alpha$. Then there is an isomorphism $\varphi': (K_1, L_1) \cong (K_2, L_2)$.

Proof. Assume the hypotheses. Let P_i, for $i = 1, 2$, be the relative algebraic closure of Q in K_i, i.e., the real algebraic numbers. Then P_i is L_i-independent, and tr.d. $K_i | P_i L_i$ = tr.d. $K_i | L_i = \aleph_\alpha$. Let φ be the "identity" isomorphism of P_1 onto P_2. Then by 16.1, φ extends to an isomorphism $\varphi': (K_1, L_1) \cong (K_2, L_2)$.

CHAPTER III

THE ELEMENTARY THEORY OF PAIRS OF REAL-CLOSED FIELDS

Section 17. Definition of the Elementary Theory

17.1. We assume familiarity with the conventional axiomatization of the class of real-closed fields, as the class of those ordered fields in which every positive element has a square root, and in which every polynomial of odd degree has a root. It is, of course, clear that in this axiomatization we could dispense with the order relation, and add extra axioms saying that any element is either a square or the negative of a square, and the set of non-zero squares is a semi-group under addition.

17.2. We are going to work with an applied first-order predicate logic with identity. This logic, $\mathcal{L}$, has the usual symbols $=, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, \forall, \exists$, and individual variables $x_0, x_1, \ldots, y_0, y_1, \ldots, z_0, z_1, \ldots$. (The above logical symbols are to be given the usual interpretation.) In addition, $\mathcal{L}$ has individual constants 0, 1, binary operation-symbols + and $\cdot$, a binary relation-symbol $<$, and a unary predicate symbol L.

A model for $\mathcal{L}$ is a 7-tuple $\langle K, L, +, \cdot, 0, 1, <\rangle$, where K is a non-empty set, L is a subset of K, + and $\cdot$ are binary operations from $K \times K$ into K, $<$ is a binary relation on K, and 0 and 1 are elements of K. No confusion should arise from our use of the same names for symbols of $\mathcal{L}$, and for the interpretation of those symbols in a model of $\mathcal{L}$. We will often identify a model $\langle K, L, +, \cdot, 0, 1, <\rangle$ with its domain K.

We assume familiarity with such fundamental concepts of model-theory as satisfaction and elementary equivalence. If M_1 and M_2 are models of $\mathcal{L}$ then we write $M_1 \equiv M_2$ to mean that M_1 is elementarily equivalent to M_2, i.e., that M_1 and M_2 satisfy the same sentences of $\mathcal{L}$.

Finally we assume we have some definite recursive definition of the relativization to L of an arbitrary formula φ of $\mathcal{L}$. If φ is an arbitrary formula of $\mathcal{L}$, then φ^L is to be the relativization to L of φ.

17.3. We are going to consider models $\langle K, L, +, \cdot, 0, 1, <\rangle$ of $\mathcal{L}$ that are pairs of real-closed fields, in the sense that $\langle K, +, \cdot, 0, 1, <\rangle$ is a real-closed field with addition $+$, multiplication $\cdot$, zero 0, unit element 1, and order $<$, and L is (the domain of) a real-closed subfield of $\langle K, +, \cdot, 0, 1, <\rangle$.

Let $\mathfrak{M}$ be the class of models of $\mathcal{L}$ that are pairs of real-closed fields. It is easy to see that $\mathfrak{M}$ may be characterized as the class of models in which a certain set of sentences of $\mathcal{L}$ is satisfied. Fix a definite recursive set A_o of sentences of $\mathcal{L}$, not involving L, which expresses that $\langle K, +, \cdot, 0, 1, <\rangle$ is a real-closed field. Let A_o^L be the set of sentences $\{\varphi^L : \varphi \in A_o\}$. Then $A_o \cup A_o^L$ is a recursive set of axioms, which is satisfied in a model $\langle K, L, +, \cdot, 0, 1, <\rangle$ if and only if the model is a pair of real-closed fields. In the remainder of this paper, $\mathfrak{A}$ is, by definition, $A_o \cup A_o^L$, and T is, by definition, the set of logical consequences of $\mathfrak{A}$. Then T is the elementary theory of pairs of real-closed fields.

17.4. The following interesting result is simply a restatement of A. Robinson's theorem that the theory of real-closed fields is model-complete.

Theorem 17.5. Let $\varphi(x_o,\ldots,x_k)$ be a formula of $\mathcal{L}$ in which L does not occur, and in which $x_o, \ldots, x_k$ are the only free variables. Then φ^* is in T, where φ^* is

$$(\forall x_o) \cdots (\forall x_k)[(Lx_o \wedge \cdots \wedge Lx_k \wedge \varphi(x_o,\ldots,x_k)) \rightarrow \varphi^L(x_o,\ldots,x_k)]$$

Section 18. The Principal Positive Results

18.1. In Chapter I we introduced, for any pair (K,L) of real-closed fields, the valuation v^L, the valuation-ring V^L, the maximal ideal I^L, the residue-class field $\mathcal{R}(L)$, and the canonical place π^L. Now, in order to avoid confusion, we modify our notation. For a given pair (K,L), let v_K^L, V_K^L, I_K^L, $\mathcal{R}_K(L)$ and π_K^L be respectively v^L, V^L, I^L, $\mathcal{R}(L)$ and π^L.

18.2. In Section 6, we showed that if (K,L) is a pair of real-closed fields, then $(\pi_K^L[L], \mathcal{R}_K(L))$ is a pair of real-closed fields, and $\pi_K^L[L]$ is cofinal in $\mathcal{R}_K(L)$. We define $\mathrm{Res}[(K,L)]$ as $(\pi_K^L[L], \mathcal{R}_K(L))$. Res is a map from models of $\mathcal{A}$ to models of $\mathcal{A}$. It is clear that if $(K_1,L_1) \cong (K_2,L_2)$, then $\mathrm{Res}[(K_1,L_1)] \cong \mathrm{Res}[(K_2,L_2)]$. The converse is not true. Let (K_1,L_1) be $(\mathbb{R}, \mathbb{R})$, and let (K_2,L_2) be $(\mathbb{R}((t^Q)), \mathbb{R})$. Then clearly (K_1,L_1) is not isomorphic to (K_2,L_2). It is, however, clear that

$$\mathrm{Res}[(K_1,L_1)] \cong \mathrm{Res}[(K_2,L_2)] \cong (\mathbb{R}, \mathbb{R}) .$$

Let $\mathcal{M}$ be the class of models of $\mathcal{A}$. Let $\mathcal{M}_o$ be the subclass of $\mathcal{M}$ consisting of those models (K,L) such that L is cofinal in K. Let $\mathcal{M}_1$ be the subclass of $\mathcal{M}$ consisting of the models (K,L) such that L is not cofinal in K. It is clear that there is a sentence ψ of $\mathcal{L}$, such that $\mathcal{M}_o$ is the class of models of $\mathcal{A} \cup \{\psi\}$, while $\mathcal{M}_1$ is the class of models of $\mathcal{A} \cup \{\ \psi\}$.

Lemma 18.3. a) If $(K,L) \in \mathcal{M}$, $\mathrm{Res}[(K,L)] \in \mathcal{M}_o$.

b) If $(K,L) \in \mathcal{M}_o$, then $\mathrm{Res}[(K,L)] \cong (K,L)$.

c) If $(K_o,L_o) \in \mathfrak{M}_o$, there is a (K_1,L_1) in $\mathfrak{M}_1$ such that $\mathrm{Res}[(K_1,L_1)] \cong (K_o,L_o)$.

d) If (K_o,L_o), (K_o',L_o') are in $\mathfrak{M}_o$, and $\mathrm{Res}[(K_o,L_o)] \cong \mathrm{Res}[(K_o',L_o')]$, then $(K_o,L_o) \cong (K_o',L_o')$.

e) There are (K_1,L_1), (K_1',L_1') in $\mathfrak{M}_1$ with $\mathrm{Res}[(K_1,L_1)] \cong \mathrm{Res}[(K_1',L_1')]$, but such that (K_1,L_1) is not isomorphic to (K_1',L_1').

Proof. a). Obvious.

b) Obvious.

c) Suppose $(K_o,L_o) \in \mathfrak{M}_o$. Let (K_1,L_1) be $(K_o((t^Q)), L_o)$. Then clearly (K_1,L_1) is in $\mathfrak{M}_1$, and $\mathrm{Res}[(K_1,L_1)] \cong (K_o,L_o)$.

d) Obvious, by (b).

e) Let (K_1,L_1) be $(\mathbb{R}((t^Q)), \mathbb{R})$, and let (K_1',L_1') be $(\mathbb{R}((t^{\mathbb{R}})), \mathbb{R})$. Then clearly (K_1,L_1) is not isomorphic to (K_1',L_1'), but $\mathrm{Res}[(K_1,L_1)] \cong \mathrm{Res}[(K_1',L_1')] \cong (\mathbb{R},\mathbb{R})$.

Recall that $\equiv$ is elementary equivalence, with respect to $\mathcal{L}$.

Lemma 18.4. If $(K_1,L_1) \equiv (K_2,L_2)$, then $\mathrm{Res}[(K_1,L_1)] \equiv \mathrm{Res}[(K_2,L_2)]$.

Proof. By 5.14, V_K^L and I_K^L are definable, in an arbitrary model (K,L), by means of formulae of $\mathcal{L}$. It follows that we may interpret in $\mathcal{L}$ the elementary theory of $\mathcal{R}_K(L)$ and π_K^L. The lemma follows.

Theorem 18.5. If (K_1,L_1) and (K_2,L_2) are in $\mathfrak{M}_1$, and $\mathrm{Res}[(K_1,L_1)] \equiv \mathrm{Res}[(K_2,L_2)]$, then $(K_1,L_1) \equiv (K_2,L_2)$.

Proof. In this proof it is convenient to assume $2^{\aleph_0} = \aleph_1$. Later we indicate how to dispense with this assumption.

We use the method of ultrapowers, and assume the basic facts about the ultrapower construction in model theory. In addition we assume Facts 1, 2, 3 listed below. In each of them, I is a countable index set, and $\mathfrak{D}$ is a non-principal ultrafilter on I.

Fact 1. (Assuming $2^{\aleph_0} = \aleph_1$). If $\mathcal{S}_1$, $\mathcal{S}_2$ are countable relational systems of the same type, with $\mathcal{S}_1$ elementarily equivalent to $\mathcal{S}_2$, then

$$\mathcal{S}_1^I/\mathfrak{D} \cong \mathcal{S}_2^I/\mathfrak{D}.$$

Fact 2. If $\mathcal{S}_1$ is a countable ordered field, then $\mathcal{S}_1^I/\mathfrak{D}$ is η_1 of cardinality $2^{\aleph_0}$.

Fact 3. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are countable fields, with $\mathcal{S}_1 \subset \mathcal{S}_2$, such that for every positive integer n there is an element x in $\mathcal{S}_2$ which is not algebraic of degree at most n over $\mathcal{S}_1$, then tr.d. $\mathcal{S}_2^I/\mathfrak{D} \mid \mathcal{S}_1^I/\mathfrak{D} = 2^{\aleph_0}$.

These facts are well-known, and in the notes we give references for them.

To prove our theorem, it clearly suffices to prove its statement for countable pairs (K_1,L_1) and (K_2,L_2), because of the Lowenheim-Skolem theorem.

Suppose, then, that (K_1,L_1) and (K_2,L_2) are countable models in $\mathfrak{M}_1$, such that $\mathrm{Res}[(K_1,L_1)] \equiv \mathrm{Res}[(K_2,L_2)]$. Using 6.5, select, for $i = 1, 2$, L_i-maximal subfields M_i of K_i. Then by 6.8, for $1 = 1, 2$,

M_i is real-closed, and $\pi_{K_i}^{L_i}[M_i] = \mathcal{R}_{K_i}(L_i)$. By definition, $L_i \subset M_i$, and so $(M_i, L_i) \cong \mathrm{Res}[(K_i, L_i)]$, for $i = 1, 2$. Note also that since (K_i, L_i) is in $\mathcal{M}_1$, $K_i \neq M_i$, and so, since K_i and M_i are real-closed, tr.d. $K_i | M_i \geq 1$.

We observe that the condition that M_i be L_i-maximal in K_i is elementary, since it is equivalent to the statement that every L_i-bounded element of K_i is L_i-infinitesimally close to an element of M_i, and every element of M_i is L_i-bounded.

Let I be a countable index set, and let $\mathcal{D}$ be a non-principal ultrafilter on I. (Such $\mathcal{D}$ exist, by the axiom of choice).

Consider, for $i = 1, 2$, the triples $(K_i^I/\mathcal{D}, M_i^I/\mathcal{D}, L_i^I/\mathcal{D})$. Concerning these triples we have the following, assuming $2^{\aleph_0} = \aleph_1$.

(a) $(K_i^I/\mathcal{D}, M_i^I/\mathcal{D}, L_i^I/\mathcal{D})$ is a triple of real-closed fields.

(b) $M_i^I/\mathcal{D}$ is $L_i^I/\mathcal{D}$-maximal in $K_i^I/\mathcal{D}$.

(c) $K_i^I/\mathcal{D}$ is η_1, of cardinality $\aleph_1$.

(d) tr.d. $K_i^I/\mathcal{D} \,|\, M_i^I/\mathcal{D} = \aleph_1$.

(e) There exists an isomorphism $\varphi: (M_1^I/\mathcal{D}, L_1^I/\mathcal{D}) \cong (M_2^I/\mathcal{D}, L_2^I/\mathcal{D})$.

(a) is clear, by basic properties of the ultrapower construction.

(b) follows from our discussion of the elementary nature of L-maximality, and the basic properties of the ultrapower construction.

(c) follows from Fact 2, and the continuum hypothesis.

(d) follows from the fact that tr.d. $K_i | M_i \geq 1$, Fact 3, and the continuum hypothesis.

(e) is seen as follows. Since $\mathrm{Res}[(K_1, L_1)] \equiv \mathrm{Res}[(K_2, L_2)]$, and for $i = 1, 2$, $\mathrm{Res}[(K_i, L_i)] \cong (M_i, L_i)$, it follows that $(M_1, L_1) \equiv (M_2, L_2)$. By Fact 1, (e) follows.

Since (a), (b), (c), (d), (e) hold, we are in a position to apply 15.1. Since $M_i^I/\mathcal{D}$ is $L_i^I/\mathcal{D}$-maximal in $K_i^I/\mathcal{D}$, it follows easily that $M_i^I/\mathcal{D}$ is $M_i^I/\mathcal{D}$-maximal in $K_i^I/\mathcal{D}$. It now follows that the map φ of (e) extends to an isomorphism

$$\varphi' : (K_1^I/\mathcal{D},\ M_1^I/\mathcal{D}) \cong (K_2^I/\mathcal{D},\ M_2^I/\mathcal{D}) .$$

It is clear that φ' maps $L_1^I/\mathcal{D}$ onto $L_2^I/\mathcal{D}$, since φ' extends φ.

Thus (K_1,L_1) and (K_2,L_2) have isomorphic ultrapowers, and so $(K_1,L_1) \equiv (K_2,L_2)$. This concludes the proof.

18.6. In order to eliminate the continuum hypothesis from the preceding proof, we give a reformulation of 18.5 as a relative completeness theorem.

Let $\mathcal{L}_1$ be an applied predicate calculus, with the same vocabulary as $\mathcal{L}$, with one of the standard recursive sets of logical axioms and rules, and having in addition the axioms $\mathcal{A}$ for pairs of real-closed fields. We assume that the syntax of $\mathcal{L}_1$ is arithmetized in some definite fashion.

In the proof of 18.4, we said that we may interpret within a model (K,L) the system $\mathrm{Res}[(K,L)]$. At that point it was unnecessary to elaborate, since Lemma 18.4 is almost transparent. Now we supply a little more detail. We claim there is a recursive map F from sentences of $\mathcal{L}$ to sentences of $\mathcal{L}$, such that for all φ_1, φ_2, $F(\varphi_1 \rightarrow \varphi_2) = F(\varphi_1) \rightarrow F(\varphi_2)$, and such that for any pair (K,L) of real-closed fields, $F(\varphi_1)$ holds in (K,L) if and only if φ_1 holds in $\mathrm{Res}[(K,L)]$. The existence of such a map is clear from the meaning of $\mathcal{A}_K(L)$. It is easy, but tedious, to give a recursive definition of such an F. We feel justified in omitting the details. Let F be a fixed map with the above properties.

Let ψ be a fixed sentence of $\mathcal{L}$ which holds precisely in models of class $\mathfrak{M}_0$. Consider the class $\mathcal{K}$ of consistent sets Δ of sentences in $\mathcal{L}_1$ such that $\mathcal{Q} \cup \{\neg\psi\} \subseteq \Delta$. Then $\Delta \in \mathcal{K}$ only if Δ has a model (K,L) such that L is not cofinal in K. For any Δ in $\mathcal{K}$ we define $\mu(\Delta)$ as $\{\varphi \mid F(\varphi)$ is provable from $\Delta\}$. It is clear that μ is arithmetical. It is clear that $\mu(\Delta)$ is consistent, for if (K,L) is a model of Δ then $\mathrm{Res}[(K,L)]$ is a model of $\mu(\Delta)$. Moreover, if Δ is closed under deduction, then so is $\mu(\Delta)$. For, suppose φ_1 and $\varphi_1 \to \varphi_2$ are in $\mu(\Delta)$. Then $F(\varphi_1)$ and $F(\varphi_1) \to F(\varphi_2)$ are in Δ.

Let us recall the syntactic notion of completeness. A set X of sentences of $\mathcal{L}$ is complete if and only if for any sentence φ of $\mathcal{L}$ either $\varphi \in X$ or $\neg\varphi \in X$.

If $\Delta \in \mathcal{K}$ and Δ is complete, then $\mu(\Delta)$ is complete. To prove this, suppose Δ is complete. By definition of $\mathcal{K}$, Δ is consistent, and so by Gödel's completeness theorem it follows that there is a model (K,L) such that Δ is precisely the set of sentences holding in (K,L). Suppose φ is an arbitrary sentence. Then either φ or $\neg\varphi$ holds in $\mathrm{Res}[(K,L)]$, and so either $F(\varphi)$ or $F(\neg\varphi)$ holds in (K,L). Thus either $F(\varphi)$ or $F(\neg\varphi)$ is in Δ, and so either φ or $\neg\varphi$ is in $\mu(\Delta)$. Therefore $\mu(\Delta)$ is complete.

For any set X of sentences of $\mathcal{L}$, let $\mathrm{Th}(X)$ be the set of all sentences provable from X.

We claim now that 18.5 is equivalent to 18.7 below.

(18.7). If $\Delta \in \mathcal{K}$, and $Th(\mu(\Delta))$ is complete, then $Th(\Delta)$ is complete.

We prove the equivalence of 18.5 and 18.7 without the use of the continuum hypothesis.

Assume 18.5, and suppose $\Delta \in \mathcal{K}$, and $Th(\mu(\Delta))$ is complete. Suppose $Th(\Delta)$ is not complete. Since Δ is consistent, $Th(\Delta)$ is consistent. Thus there are models (K_1,L_1) and (K_2,L_2) of $Th(\Delta)$, such that (K_1,L_1) is not elementarily equivalent to (K_2,L_2). Since $\Delta \in \mathcal{K}$, (K_i,L_i) is in $\mathcal{M}_1$, for $i = 1, 2$. By 18.5 it follows that $Res[(K_1,L_1)]$ is not elementarily equivalent to $Res[(K_2,L_2)]$. But if $\varphi \in \mu(\Delta)$, $F(\varphi)$ is provable from Δ, and so $F(\varphi)$ holds in both (K_1,L_1) and (K_2,L_2), and so φ holds in both $Res[(K_1,L_1)]$ and $Res[(K_2,L_2)]$. Thus $Res[(K_1,L_1)]$ and $Res[(K_2,L_2)]$ are models of the complete set $Th(\mu(\Delta))$, and so are elementarily equivalent. This gives a contradiction. It follows that 18.5 implies 18.7.

For the converse, assume 18.7. Suppose (K_1,L_1) and (K_2,K_2) are in $\mathcal{M}_1$, and $Res[(K_1,L_1)] \equiv Res[(K_2,L_2)]$. Let X be the set of all sentences holding in $Res[(K_1,L_1)]$. Then X is the set of all sentences holding in $Res[(K_2,L_2)]$. Let Δ be $F[X] \cup \mathcal{a} \cup \{\neg\psi\}$. Then $\Delta \in \mathcal{K}$, and Δ is satisfied in both (K_1,L_1) and (K_2,L_2). But $\mu(\Delta) = X$, and X is complete. By 18.7, $Th(\Delta)$ is complete, and so $(K_1,L_1) \equiv (K_2,L_2)$. It follows that 18.7 implies 18.5.

Now in terms of the arithmetization of syntax of $\mathcal{L}_1$, 18.7 is arithmetical in the parameter Δ. We now know that 18.7 is provable in set theory, using the axiom of choice and the continuum hypothesis. By working in the universe of sets constructible from Δ, we may eliminate the use of the axiom of choice and the continuum hypothesis.

Thus 18.5 does not depend on the continuum hypothesis.

18.8. Consider the following four classes A, B, C, D of models of $\mathfrak{a}$.

Class A. This class consists of all pairs (K,L) such that K = L.

Class B. This class consists of all pairs (K,L) such that $K \neq L$ and L is dense in K.

Class C. This class consists of all pairs (K,L) such that L is not cofinal in K and Res[(K,L)] is in A.

Class D. This class consists of all pairs (K,L) such that L is not cofinal in K and Res[(K,L)] is in B.

Each of the above classes is non-empty. $(\mathbb{R},\mathbb{R})$ is in A, $(\mathbb{R},\tilde{\mathbb{Q}})$ is in B, where $\tilde{\mathbb{Q}}$ is the field of real algebraic numbers, $(\mathbb{R}((t^{\mathbb{Q}})),\mathbb{R})$ is in C, and $(\mathbb{R}((t^{\mathbb{Q}})),\tilde{\mathbb{Q}})$ is in D.

It is obvious that each of the classes A, B, C, D has a recursive axiomatization.

Theorem 18.9. Let J be one of the classes A, B, C or D. Suppose (K_1,L_1) and (K_2,L_2) are in J. Then $(K_1,L_1) \equiv (K_2,L_2)$.

Proof. If J = A, this is no more than a restatement of Tarski's classical theorem.

If J = B, this is a theorem of A. Robinson. We will prove this theorem later.

If J = C, the result follows from 18.5, and the case J = A.

If J = D, the result follows from 18.5, and the case J = B.

Corollary 18.10. Let J be one of the classes A, B, C, D. Then J is decidable, i.e., the set of sentences holding in all numbers of J is recursive.

Proof. This is an easy consequence of the fact that J has a recursive, complete axiomatization.

Corollary 18.11. There is a decision procedure for determining whether an arbitrary sentence φ holds in all pairs (K,L) of real-closed fields such that either K or L is archimedean.

Proof. a) Suppose (K,L) is a pair of real-closed fields, where K is archimedean. Then, without loss of generality, K and L are subfields of $\mathbb{R}$, by a classical result. Then clearly $K = L$ or $K \neq L$ and L is dense in K.

b) Suppose (K,L) is a pair of real-closed fields, where L is archimedean. If L is cofinal in K, then K is archimedean, and we have already discussed this. In any case L is cofinal in $\mathcal{R}_K(L)$, and so $\mathcal{R}_K(L)$ is archimedean. Then $L = \mathcal{R}_K(L)$, or L is dense in $\mathcal{R}_K(L)$.

Thus if (K,L) is a pair of real-closed fields, one of which is archimedean, then (K,L) is in $A \cup B \cup C \cup D$. On the other hand, as we showed by examples, each of the classes A, B, C, D contains a pair (K,L) such that either K or L is archimedean.

The result follows by 18.9 and 18.10.

18.12. From results about elementary equivalence, we now turn to results about elementary extension. (We assume the reader is familiar with the relation of elementary extension, written $\prec$).

Suppose (K_1,L_1) and (K_2,L_2) are models of $\mathcal{A}$. Then, by definition, $(K_1,L_1) \subseteq (K_2,L_2)$ if and only if K_1 is a subfield of K_2 and $K_1 \cap L_2 = L_1$. We are interested in finding conditions under which $(K_1,L_1) \prec (K_2,L_2)$. Our main result in this direction is an analogue of 18.5. We now prove the basic result from which the main result will follow.

Theorem 18.13. Suppose (K_1,L_1) and (K_2,L_2) are pairs of real-closed fields, such that $(K_1,L_1) \subseteq (K_2,L_2)$, L_1 is not cofinal in K_1 and L_2 is not cofinal in K_2. Suppose M_1, M_2 are subfields of K_1, K_2 respectively such that the following conditions hold.

1. $V_{K_2}^{L_2} \cap K_1 = V_{K_1}^{L_1}$.

2. M_1 is L_1-maximal in K_1.

3. M_2 is L_2-maximal in K_2.

4. $(M_1,L_1) \prec (M_2,L_2)$.

Then $(K_1,L_1) \prec (M_2,L_2)$.

Proof. We prove the result by the method of ultrapowers. We will use the continuum hypothesis, but this may be eliminated, as we indicate at the end.

Recall the proof of 18.5. There we stated without proof certain important facts about the ultrapower construction. For the present proof we need almost the same facts, the only difference being that Fact 1 is to be replaced by the following Fact 1'. (As usual, I is a countable index set and $\mathfrak{D}$ is a non-principal ultrafilter on I.)

Fact 1'. (Assuming $2^{\aleph_0} = \aleph_1$). If $\mathcal{S}_1$, $\mathcal{S}_2$ are countable relational systems, with $\mathcal{S}_1 \prec \mathcal{S}_2$, then the injection $i: \mathcal{S}_1 \to \mathcal{S}_2$ extends to an isomorphism of $\mathcal{S}_1^I/\mathfrak{D}$ onto $\mathcal{S}_2^I/\mathfrak{D}$.

In addition to this rather profound fact, we make constant use of the elementary fact that if $\mathcal{S}$ is a relational system then $\mathcal{S} \prec \mathcal{S}^I/\mathfrak{D}$.

As a preliminary to the proof of 18.13 we show that it suffices to prove the result for countable pairs (K_1,L_1) and (K_2,L_2). For, suppose we have proved the result for countable pairs. Let K_1, L_1, M_1, K_2, L_2, M_2 satisfy the hypotheses of the theorem. Let $x_1, \ldots, x_n$ be (finitely many) elements of K_1. By the Löwenheim-Skolem theorem there is a countable subfield $\hat{K}_1$ of K_1 such that $\{x_1, \ldots, x_n\} \subseteq \hat{K}_1$ and $(\hat{K}_1, \hat{K}_1 \cap M_1, \hat{K}_1 \cap L_1) \prec (K_1, M_1, L_1)$. Similarly, there is a countable subfield $\hat{K}_2$ of K_2 such that $\hat{K}_1 \subseteq \hat{K}_2$ and $(\hat{K}_2, \hat{K}_2 \cap M_2, \hat{K}_2 \cap L_2) \prec (K_2, M_2, L_2)$. Then clearly $\hat{K}_1 \cap L_1$ is not cofinal in $\hat{K}_1$, and $\hat{K}_2 \cap L_2$ is not cofinal in $\hat{K}_2$. If one recalls the remarks in 18.5 on the elementary character of maximality, one sees that $\hat{K}_1 \cap M_1$ is $(\hat{K}_1 \cap L_1)$-maximal in $\hat{K}_1$, and $\hat{K}_2 \cap M_2$ is $(\hat{K}_2 \cap L_2)$-maximal in K_2. We leave to the reader the exercises of showing that

$$V_{\hat{K}_2}^{\hat{K}_2 \cap L_2} \cap \hat{K}_1 = V_{\hat{K}_1}^{\hat{K}_1 \cap L_1} ;$$

and that

$$(\hat{K}_1 \cap M_1, \hat{K}_1 \cap L_1) \prec (\hat{K}_2 \cap M_2, \hat{K}_2 \cap L_2) .$$

It follows by the countable case of 18.13 that

$$(\hat{K}_1, \hat{K}_1 \cap L_1) \prec (\hat{K}_2, \hat{K}_2 \cap L_2)$$

Now suppose $\varphi(x_1,\ldots,x_n)$ is a sentence of $\mathcal{L}$, with parameters $x_1, \ldots, x_n$, such that $\varphi(x_1,\ldots,x_n)$ holds in (K_2, L_2). Since $(\hat{K}_2, \hat{K}_2 \cap L_2) \prec (K_2, L_2)$, $\varphi(x_1,\ldots,x_n)$ holds in $(\hat{K}_2, \hat{K}_2 \cap L_2)$. Since $(\hat{K}_1, \hat{K}_1 \cap L_1) \prec (\hat{K}_2, \hat{K}_2 \cap L_2)$, $\varphi(x_1,\ldots,x_n)$ holds in $(\hat{K}_1, \hat{K}_1 \cap L_1)$.

Finally, since $(K_1, K_1 \cap L_1) \prec (K_1, L_1)$, $\varphi(x_1, \ldots, x_n)$ holds in (K_1, L_1). Since $x_1, \ldots, x_n$ were chosen arbitrarily from K_1, we conclude that $(K_1 L_1) \prec (K_2, L_2)$. Thus the general case of 18.13 follows from the countable case, and henceforward we confine ourselves to the countable case.

Thus, let K_1, L_1, M_1, K_2, L_2, M_2 be countable real-closed fields satisfying the hypotheses of 18.13. Let I be a countable index set and $\mathfrak{D}$ a non-principal ultrafilter on I. (Such $\mathfrak{D}$ exist by the axiom of choice). Our objective is to show that the injection $i: (K_1, L_1) \to (K_2, L_2)$ extends to an isomorphism of $(K_1^I/\mathfrak{D}, L_1^I/\mathfrak{D})$ onto $(K_2^I/\mathfrak{D}, L_2^I/\mathfrak{D})$. From this result, and the standard facts about the ultrapower construction, it will clearly follow that $(K_1, L_1) \prec (K_2, L_2)$.

We first observe that K_1 and M_2 are linearly disjoint over M_1. For suppose $\mu_1, \ldots, \mu_n$ are in M_2 and linearly dependent over K_1. Then there are $x_1, \ldots, x_n$ in K_1, not all zero such that $x_1 \cdot \mu_1 + \cdots + x_n \cdot \mu_n = 0$. We may suppose, without loss of generality, that, for $1 \le j \le n$, $|x_j| \le |x_1|$. We define, for $1 \le j \le n$, y_j as $x_j \cdot x_1^{-1}$. Then $y_1 = 1$, and, for $1 \le j \le n$, $y_j \in V_{K_1}^{L_1}$. Since M_1 is L_1-maximal in K_1, there are elements m_j of M_1 such that, for $1 \le j \le n$, $y_j - m_j$ is L_1-infinitesimal. Clearly $m_1 = 1$. Now,

$$y_1 \cdot \mu_1 + \cdots + y_n \cdot \mu_n = 0 ,$$

and so

$$(y_1 - m_1) \cdot \mu_1 + \cdots + (y_n - m_n)\mu_n = -m_1 \cdot \mu_1 - \cdots - m_n \cdot \mu_n \in M_2 .$$

Since $y_j - m_j$ is L_1-infinitesimal, $y_j - m_j$ is M_2-infinitesimal.

Therefore $(y_1 - m_1) \cdot \mu_1 + \cdots + (y_n - m_n) \cdot \mu_n$ is M_2-infinitesimal. But

$$(y_1 - m_1) \cdot \mu_1 + \cdots + (y_n - m_n) \cdot \mu_n \in M_2 ,$$

so

$$(y_1 - m_1) \cdot \mu_1 + \cdots + (y_n - m_n) \cdot \mu_n = 0 .$$

Therefore $m_1 \cdot \mu_1 + \cdots + m_n \cdot \mu_n = 0$, and since $m_1 = 1$, it follows that $\mu_1, \ldots, \mu_n$ are linearly dependent over M_1. Since $M_1 = K_1 \cap M_2$, by conditions 1, 2, 3, it follows that K_1 and M_2 are linearly disjoint over their intersection M_1.

We next observe that, by Fact 1' and the assumption that $(M_1, L_1) \prec (M_2, L_2)$, the injection $j: (M_1, L_1) \to (M_2, L_2)$ extends to an isomorphism φ_0 of $(M_1^I/\mathfrak{D}, L_1^I/\mathfrak{D})$ onto $(M_2^I/\mathfrak{D}, L_2^I/\mathfrak{D})$. Now, let B be a transcendence base for K_1 over M_1. By basic properties of the ultrapower construction, B is algebraically independent over $M_1^I/\mathfrak{D}$, so that B is a transcendence base for $K_1 \cdot M_1^I/\mathfrak{D}$ over $M_1^I/\mathfrak{D}$. Since K_1 and M_2 are linearly disjoint over M_1, B is algebraically independent over M_2, and so over $M_2^I/\mathfrak{D}$, whence B is a transcendence base for $K_1 \cdot M_2^I/\mathfrak{D}$ over $M_2^I/\mathfrak{D}$. By the Steinitz theory, φ_0 has a unique extension to a field-isomorphism φ_1 of $M_1^I/\mathfrak{D}(B)$ onto $M_2^I/\mathfrak{D}(B)$, such that φ_1 is the identity on B. Then φ_1 is the identity on $M_1(B)$.

We now show that φ_1 is order-preserving. It suffices to show that φ_1 is order-preserving on $M_1^I/\mathfrak{D}[B]$. This will follow from (*) below.

(*) (n a positive integer). If $\mu_1, \ldots, \mu_n \in M_1^I/\mathfrak{D}$, and $t_1, \ldots, t_n \in M_1(B)$, and $\mu_1 \cdot t_1 + \cdots + \mu_n \cdot t_n > 0$, then $\varphi_o(\mu_1) \cdot t_1 + \cdots + \varphi_o(\mu_n) \cdot t_n > 0$.

We prove (*) by induction on n. If $n = 1$, the result is clear. Suppose we have proved the result for all $n < N$. Suppose $\mu_1, \ldots, \mu_N \in M_1^I/\mathfrak{D}$, $t_1, \ldots, t_N \in M_1(B)$, and $\mu_1 \cdot t_1 + \cdots + \mu_N \cdot t_N > 0$. Select j with $1 \leq j \leq N$ such that $|t_j|$ is maximal among $|t_1|, \ldots, |t_N|$. Then $|t_j| > 0$. For $1 \leq n \leq N$, define t_n' as $t_n \cdot |t_j|^{-1}$. Then t_j' is 1 or -1, and, for $1 \leq n \leq N$, t_n' is in $V_{K_1}^{L_1}$. Since M_1 is L_1-maximal in K_1, there are, for $1 \leq n \leq N$, u_n in M_1 such that $t_n' - u_n$ is L_1-infinitesimal. Note that $u_j = t_j'$. By basic properties of ultrapowers, $t_n' - u_n$ is $L_1^I/\mathfrak{D}$-infinitesimal in $K_1^I/\mathfrak{D}$. Also, since $V_{K_2}^{L_2} \cap K_1 = V_{K_1}^{L_1}$, $t_n' - u_n$ is L_2-infinitesimal in K_2, and so $L_2^I/\mathfrak{D}$-infinitesimal in $K_2^I/\mathfrak{D}$. Define ε_n as $t_n' - u_n$. Since $\mu_1 \cdot t_1 + \cdots + \mu_N \cdot t_N > 0$, $\mu_1 \cdot t_1' + \cdots + \mu_N \cdot t_N' > 0$. Thus $\mu_1 \cdot (u_1+\varepsilon_1) + \cdots + \mu_N \cdot (u_N+\varepsilon_N) > 0$. Now, $\mu_1 \cdot \varepsilon_1 + \cdots + \mu_N \cdot \varepsilon_N$ is $L_1^I/\mathfrak{D}$-infinitesimal in $K_1^I/\mathfrak{D}$, and $\mu_1 \cdot u_1 + \cdots + \mu_N \cdot u_N$ is not $L_1^I/\mathfrak{D}$-infinitesimal, unless $\mu_1 \cdot u_1 + \cdots + \mu_N \cdot u_N = 0$. We distinguish two cases.

Case 1. $\mu_1 \cdot u_1 + \cdots + \mu_N \cdot u_N \neq 0$. Then clearly $\mu_1 \cdot u_1 + \cdots + \mu_N \cdot u_N > 0$. Since φ_o is an order-isomorphism and φ_o is the identity on M_1, $\varphi_o(\mu_1) \cdot u_1 + \cdots + \varphi_o(\mu_N) \cdot u_N > 0$. Evidently $\varphi_1(\mu_1) \cdot u_1 + \cdots + \varphi_o(\mu_N) \cdot u_N$ is not $L_2^I/\mathfrak{D}$-infinitesimal, whereas $\varphi_o(\mu_1) \cdot \varepsilon_1 + \cdots + \varphi_o(\mu_N) \cdot \varepsilon_N$ is $L_2^I/\mathfrak{D}$-infinitesimal. Therefore $\varphi_o(\mu_1) \cdot t_1' + \cdots + \varphi_o(\mu_N) \cdot t_N' > 0$

Case 2. $\mu_1 \cdot u_1 + \cdots + \mu_N \cdot u_N = 0$. Then $\varphi_o(\mu_1) \cdot u_1 + \cdots + \varphi_o(\mu_N) \cdot u_N = 0$. Also, $\mu_1 \cdot \varepsilon_1 + \cdots + \mu_N \cdot \varepsilon_N > 0$. Recall that $\varepsilon_j = 0$. Then, by the induction hypothesis,

$$\varphi_o(\mu_1) \cdot \varepsilon_1 + \cdots + \varphi_o(\mu_N) \cdot \varepsilon_N > 0 ,$$

so that

$$\varphi_o(\mu_1) \cdot t_1' + \cdots + \varphi_o(\mu_N) \cdot t_N' > 0 .$$

Thus, in all cases,

$$\varphi_o(\mu_1) \cdot t_1 + \cdots + \varphi_o(\mu_N) \cdot t_N > 0 .$$

This completes the inductive step for (*), and so (*) is proved. Thus φ_1 is order-preserving.

By 1.8, φ_1 has a unique extension to an isomorphism φ_2 of $\mathrm{Re}(K_1 \cdot M_1^I/\mathcal{D})$ onto $\mathrm{Re}(K_1 \cdot M_2^I/\mathcal{D})$. Clearly φ_2 is the identity on K_1.

Since K_1 is countable, tr.d. $\mathrm{Re}(K_1 \cdot M_1^I/\mathcal{D})/M_1^I/\mathcal{D} < \aleph_1$, and tr.d. $\mathrm{Re}(K_1 \cdot M_2^I/\mathcal{D})/M_2^I/\mathcal{D} < \aleph_1$. By arguments similar to those of 18.5, one sees that $K_1^I/\mathcal{D}$ and $K_2^I/\mathcal{D}$ are η_1 real-closed fields, $M_1^I/\mathcal{D}$ is $M_1^I/\mathcal{D}$-maximal in $K_1^I/\mathcal{D}$, $M_2^I/\mathcal{D}$ is $M_2^I/\mathcal{D}$-maximal in $K_2^I/\mathcal{D}$, tr.d. $K_1^I/\mathcal{D}/M_2^I/\mathcal{D} = \aleph_1$, and tr.d. $K_2^I/\mathcal{D}/M_2^I/\mathcal{D} = \aleph_1$. From 9.2, 13.4 and 15.1 it follows that φ_2 extends to an isomorphism φ_3 of $K_1^I/\mathcal{D}$ onto $K_2^I/\mathcal{D}$. Clearly φ_3 maps $L_1^I/\mathcal{D}$ onto $L_2^I/\mathcal{D}$, and extends $i\colon (K_1,L_1) \to (K_2,L_2)$. This concludes the proof of 18.13.

18.14. We now indicate how to eliminate the continuum hypotheses from the above proof. The basic idea is quite standard, in that we replace 18.13 by an equivalent syntactical result (where the equivalence is proved without the use of the continuum hypothesis). Recall that in the preceding proof we showed that 18.13 is equivalent to the "countable" version where K_1, L_1, M_1, K_2, L_2, M_2 are restricted to be countable. We now indicate how this "countable" version is equivalent to a syntactical result.

We consider two extensions $\mathcal{L}_1$ and $\mathcal{L}_2$ of the logic $\mathcal{L}$. $\mathcal{L}_1$ is obtained from $\mathcal{L}$ by adjoining individual constants $a_1, a_2, \ldots, a_n, \ldots$. $\mathcal{L}_2$ is obtained from $\mathcal{L}_1$ by adjoining a unary predicate-symbol M. Recall that $\mathcal{L}$ has exactly two individual constants, 0 and 1. If S is any set of sentences of $\mathcal{L}_2$, we define dom(S) as the set consisting of the individual constants 0, 1 and all individual constants occurring in members of S. We say S is complete if for any sentence φ, whose individual constants lie in dom(S), either φ or $\neg\varphi$ is in S. Suppose S is complete. Then we say S is existentially complete if for any sentence $(\exists x)\varphi(x)$ in S there is an individual constant C such that $\varphi(C)$ is in S. If S is existentially complete, we define diag(S) (the diagram of S) as the set of atomic sentences of S.

For sentences of $\mathcal{L}_1$ we fix a definite recursive definition of relativization to M, such that this relativization is the identity on atomic sentences. For a sentence φ of $\mathcal{L}_1$ let $\varphi^{(M)}$ be the relativization of φ to M. Then $\varphi^{(M)}$ is a sentence of $\mathcal{L}_2$. If S is a set of sentences of $\mathcal{L}_2$ we define $\lambda(S)$ as the set of all sentences φ of $\mathcal{L}_1$ such that $\varphi^{(M)} \in S$.

Let (Max) be a fixed recursive set of sentences of $\mathcal{L}_2$, (with no individual constants except 0, 1), characterizing the class of models

$\langle K, L, M, +, \cdot, 0, 1, < \rangle$ such that K is a real-closed field, L and M are real-closed subfields of K, L is not cofinal in K and M is L-maximal in K.

Let $\inf(x_1)$ be a formula of $\mathcal{L}$, with x_1 as its only free variable, which expresses that x_1 is not in V^L (i.e., x_1 is L-infinite). It is easy to devise such formulae, and there seems little point in writing one out.

We can now state the syntactical equivalent 18.15 of the "countable" version of 18.13.

<u>18.15</u>. Suppose T_1, T_2 are consistent, complete, existentially complete sets of sentences of $\mathcal{L}_2$. Suppose the following conditions hold.

1. $\mathrm{diag}(T_1) \subseteq T_2$.
2. For any individual constant C of $\mathcal{L}_2$, if $\inf(C)$ is in T_1 then $\inf(C)$ is in T_2.
3. $(\mathrm{Max}) \subseteq T_1 \cap T_2$.
4. $\lambda(T_1) \subseteq \lambda(T_2)$.

Then $T_1 \subseteq T_2$.

We will leave to the reader the exercise of showing that 18.15 is equivalent to the "countable" version of 18.13. The connection between countable models of $\mathcal{L}_2$ and consistent, complete, existentially complete sets of sentences of $\mathcal{L}_2$ ought to be clear to anyone acquainted both with Henkin's proof of Gödel's Completeness Theorem, and with A. Robinson's notion of the diagram of a model.

Now we note that 18.15 is arithmetical in the parameters T_1 and T_2. Therefore, by working in the universe of sets constructible from

T_1 and T_2, we can eliminate the continuum hypothesis from any proof (in set theory) of 18.15. Thus 18.13 does not depend on the continuum hypothesis.

18.16. We now apply 18.13 to get an analogue of 18.5. Recall the notation of 18.5. The required analogue is 18.17 below.

Theorem 18.17. Suppose (K_1,L_1) and (K_2,L_2) are in $\mathfrak{M}_1$, and $(K_1,L_1) \subseteq (K_2,L_2)$. Suppose further that $V_{K_2}^{L_2} \cap K_1 = V_{K_1}^{L_1}$. Then $\mathrm{Res}[(K_1,L_1)] \subseteq \mathrm{Res}[(K_2,L_2)]$, and if $\mathrm{Res}[(K_1,L_1)] \prec \mathrm{Res}[(K_2,L_2)]$ then $(K_1,L_1) \prec (K_2,L_2)$.

Proof. Let (K_1,L_1), (K_2,L_2) satisfy the hypothesis of 18.17. Since $V_{K_2}^{L_2} \cap K_1 = V_{K_1}^{L_1}$, it is clear that the valuation $v_{K_2}^{L_2}$ is an extension of the valuation $v_{K_1}^{L_1}$, and it follows easily that $\mathrm{Res}[(K_1,L_1)]$ is naturally embedded in $\mathrm{Res}[(K_2,L_2)]$. This proves the first part of the theorem.

Now suppose in addition that $\mathrm{Res}[(K_1,L_1)] \prec \mathrm{Res}[(K_2,L_2)]$. Let M_1 be any L_1-maximal subfield of K_1. We claim $M_1 \cdot L_2$ is a subfield of $V_{K_2}^{L_2}$. Suppose not. Then there is an integer n, and non-zero elements $m_1, \dots, m_n$ of M_1, and non-zero elements $\ell_1', \dots, \ell_n'$ of L_2 such that

$$m_1 \cdot \ell_1' + \cdots + m_n \cdot \ell_n'$$

is a non-zero L_2-infinitesimal. Clearly $n > 1$, and let us assume n is chosen minimal with the property that there exist $m_1, \dots, m_n$, $\ell_1', \dots, \ell_n'$ as above. Then

$$\pi^{L_1}(m_1) \cdot \pi^{L_2}(\ell_1') + \cdots + \pi^{L_1}(m_n) \cdot \pi^{L_2}(\ell_n') = 0 .$$

But since $\mathrm{Res}[(K_1,L_1)] \prec \mathrm{Res}[(K_2,L_2)]$, it follows that there are elements $\ell_1, \ldots, \ell_n$ in L_1 such that

$$\pi^{L_1}(m_1) \cdot \pi^{L_1}(\ell_1) + \cdots + \pi^{L_1}(m_n) \cdot \pi^{L_1}(\ell_n) = 0 .$$

Then

$$\pi^{L_1}(m_1 \cdot \ell_1 + \cdots + m_n \cdot \ell_n) = 0 .$$

Since M_1 is a subfield of V^{L_1}, with $L_1 \subseteq M_1$, it follows that $m_1 \cdot \ell_1 + \cdots + m_n \cdot \ell_n = 0$. Then

$$\ell_n \cdot (m_1 \cdot \ell_1' + \cdots + m_n \cdot \ell_n') - \ell_n' \cdot (m_1 \cdot \ell_1 + \cdots + m_n \cdot \ell_n)$$

is a non-zero L_2-infinitesimal. Then

$$m_1(\ell_n\ell_1'-\ell_n'\ell_1) + \cdots + m_{n-1}(\ell_n\ell_{n-1}'-\ell_n'\ell_{n-1})$$

is a non-zero L_2-infinitesimal, contrary to the minimality of n. It follows that $M_1 \cdot L_2$ is a subfield of $V_{K_2}^{L_2}$, and by Zorn's lemma $M_1 \cdot L_2$ extends to an L_2-maximal subfield of K_2. Let M_2 be such an extension. Then clearly $(M_1,L_1) \subseteq (M_2,L_2)$. Equally clearly, $\mathrm{Res}[(K_2,L_2)]$ is isomorphic to (M_2,L_2), by an isomorphism that maps $\mathrm{Res}[(K_1,L_1)]$ onto (M_1,L_1). It follows that $(M_1,L_1) \prec (M_2,L_2)$. By 18.13, it follows that $(K_1,L_1) \prec (K_2,L_2)$. This concludes the proof.

18.18. Recall the classes A,B,C,D of 18.8. In 18.9 we showed that if J is one of the above classes, and (K_1,L_1) and (K_2,L_2) are in J, then

$(K_1, L_1) \equiv (K_2, L_2)$. We now prove an analogous result for the notion of elementary extension. However, this result is not as neat as 18.9.

Theorem 18.20. Suppose (K_1, L_1) and (K_2, L_2) are pairs of real-closed fields, with $(K_1, L_1) \subseteq (K_2, L_2)$. Then the following results hold.

a) If (K_1, L_1) and (K_2, L_2) are in A, then $(K_1, L_1) \prec (K_2, L_2)$.

b) If (K_1, L_1) and (K_2, L_2) are in B, and K_1 and L_2 are linearly disjoint over L_1, then $(K_1, L_1) \prec (K_2, L_2)$.

c) If (K_1, L_1) and (K_2, L_2) are in C, and $V_{K_2}^{L_2} \cap K_1 = V_{K_1}^{L_1}$, then $(K_1, L_1) \prec (K_2, L_2)$.

d) If (K_1, L_1) and (K_2, L_2) are in D, and $V_{K_2}^{L_2} \cap K_1 = V_{K_1}^{L_1}$, and $\pi^{L_2}[V_{K_1}^{L_1}]$ and $\pi^{L_2}[L_2]$ are linearly disjoint over $\pi^{L_2}[L_1]$, then $(K_1, L_1) \prec (K_2, L_2)$.

Proof. a) This is simply a restatement of A. Robinson's theorem on the model-completeness of the theory of real-closed fields.

b) This is a result of A. Robinson, which we will prove shortly.

c) This follows from (a) and 18.17.

d) This follows from (b) and 18.17.

Remark. A. Robinson showed by means of an example that the restrictive clause in (b), namely that K_1 and L_2 are linearly disjoint over L_1, is necessary.

We now show that in (c) the restrictive clause, namely that $V_{K_2}^{L_2} \cap K_1 = V_{K_1}^{L_1}$, is necessary. Let L_1 be $\mathbb{R}$. Let t be transcendental over $\mathbb{R}$, and order $\mathbb{R}(t)$ so that $t > \mathbb{R}$. Let K_1 be the real-closure of $\mathbb{R}(t)$. Then clearly (K_1,L_1) is in C. Let u be transcendental over K_1, and order $K_1(u)$ so that $u > K_1$. Let K_2 be the real-closure of $K_1(u)$. Let L_2 be the real-closure of $\mathbb{R}(t+1/u)$. Then it is an exercise for the reader to verify that (K_2,L_2) is in C, and $(K_1,L_1) \subseteq (K_2,L_2)$. We observe that $t \notin V_{K_1}^{L_1}$, whereas $t \in V_{K_2}^{L_2}$. Then clearly (K_2,L_2) is not an elementary extension of (K_1,L_1).

A similar example will show that the restrictive clause in (d) is necessary.

<u>18.21</u>. We will conclude this section by proving, via ultrapowers, Robinson's result about Class B. Before doing so, we mention an operation, on pairs of real-closed fields, which seems to us to be of interest, but about which we know almost nothing.

Let (K,L) be a pair of real-closed fields, and let $\bar{L}$ be the closure of L in K, in the sense of Section 11. Then $(K,\bar{L})$ is also a pair of real-closed fields. We define $\text{clos}[(K,L)]$ as $(K,\bar{L})$. It is quite obvious that if $(K_1,L_1) \equiv (K_2,L_2)$ then $\text{clos}[(K_1L_1)] \equiv \text{clos}[(K_2,L_2)]$. We observe that Class A consists of all pairs $\text{clos}[(K,L)]$ where (K,L) is in B.

<u>Thoerem 18.22</u>. (i) If (K_1,L_1) and (K_2,L_2) are in Class B, then $(K_1,L_1) \equiv (K_2,L_2)$

(ii) If (K_1,L_1) and (K_2,L_2) are in Class B, and $(K_1,L_1) \subseteq (K_2,L_2)$, and K_1 and L_2 are linearly disjoint over L_1, then $(K_1,L_1) \prec (K_2,L_2)$.

<u>Proof</u>. By the Löwenheim-Skolem theorem it suffices to prove (i) and (ii) subject to the restriction that K_1, L_1, K_2, L_2 are countable (Recall the proof of 18.13).

Further, we will use the continuum hypothesis in our proof. However, by the usual method we may eliminate the continuum hypothesis, because our conclusions are equivalent to syntactical results.

(i) Suppose (K_1, L_1) and (K_2, L_2) are in Class B, where K_1, L_1, K_2, L_2 are countable. Let I be a countable index set, and let $\mathfrak{D}$ be a non-principal ultrafilter on I. Then $K_1^I/\mathfrak{D}$, $L_1^I/\mathfrak{D}$, $K_2^I/\mathfrak{D}$ and $L_2^I/\mathfrak{D}$ are η_1 real-closed fields of cardinality $\aleph_1$ (assuming $2^{\aleph_0} = \aleph_1$). Further, $L_1^I/\mathfrak{D}$ is dense in $K_1^I/\mathfrak{D}$, and $L_2^I/\mathfrak{D}$ is dense in $K_2^I/\mathfrak{D}$. Finally, assuming $2^{\aleph_0} = \aleph_1$, tr.d. $K_1^I/\mathfrak{D} \,|\, L_1^I/\mathfrak{D} = \aleph_1$, and tr.d. $K_2^I/\mathfrak{D} \,|\, L_2^I/\mathfrak{D} = \aleph_1$. Therefore by 16.2, $(K_1^I/\mathfrak{D}, L_1^I/\mathfrak{D}) \cong (K_2^I/\mathfrak{D}, L_2^I/\mathfrak{D})$. We conclude that $(K_1, L_1) \equiv (K_2, L_2)$.

(ii) Let (K_1, L_1), (K_2, L_2) be as in (i) above, and suppose in addition that $(K_1, L_1) \subseteq (K_2, L_2)$, and K_1 and L_2 are linearly disjoint over L_1. Let I and $\mathfrak{D}$ be as in (i) above. By basic properties of the ultrapower construction, $K_1 \cap L_1^I/\mathfrak{D} = L_1$, and K_1 and $L_1^I/\mathfrak{D}$ are linearly disjoint over L_1. Similarly, using the facts that $(K_1, L_1) \subseteq (K_2, L_2)$ and K_1 and L_2 are linearly disjoint over L_1, one sees that $K_1 \cap L_2^I/\mathfrak{D} = L_1$, and K_1 and $L_2^I/\mathfrak{D}$ are linearly disjoint over L_1. Thus K_1 is both $L_1^I/\mathfrak{D}$-independent and $L_2^I/\mathfrak{D}$-independent. Using the facts in (i) above, one concludes by 16.1 that the injection $i: (K_1, L_1) \to (K_2, L_2)$ extends to an isomorphism of $(K_1^I/\mathfrak{D}, L_1^I/\mathfrak{D})$ onto $(K_2^I/\mathfrak{D}, L_2^I/\mathfrak{D})$. It follows that $(K_1, L_1) \prec (K_2, L_2)$.

This concludes the proof.

Section 19. Gaps

In this section, $(\mathcal{S}_1, \mathcal{S}_2)$ is either a pair of D-groups, or a pair of ordered rings. Let 0, $+$, $<$ be respectively the zero, addition and order on $\mathcal{S}_1$. Our definitions below are to be understood as relative to a given pair $(\mathcal{S}_1, \mathcal{S}_2)$.

Definition 19.1. For x, y, z in $\mathcal{S}_1$, $B(x,y,z) =_{def} (x<y<z) \vee (z<y<x)$.

Definition 19.2. For x in $\mathcal{S}_1$, $\Gamma(x) = \{t \mid (\forall y \in \mathcal{S}_2) \neg B(x,y,x+t)\}$.

Definition 19.3. $S = \{x \mid (\forall y \in \mathcal{S}_2) \neg B(x,y,2x)\}$ (Alternatively, $S = \{x \mid x \in \Gamma(x)\}$).

Lemma 19.4. a) For all x, $0 \in \Gamma(x)$.

b) For all x, $\Gamma(x)$ is convex in $\mathcal{S}_1$

c) $0 \in S$

d) For all x, if $x \in S$ then $-x \in S$.

Proof. Trivial.

Lemma 19.5. If $\Gamma(x)$ is symmetric for all x, then $S = \{0\}$.

Proof. Suppose $\Gamma(x)$ is symmetric for each x, and suppose s is a non-zero element of S. Since S is symmetric, we may assume without loss of generality that $s > 0$. Since $s \in S$ and $B(s, 3s/2, 2s)$, it follows that $3s/2 \notin \mathcal{S}_1$, and so $s \notin \mathcal{S}_1$. Then $2s \notin \mathcal{S}_1$. Now, since there is no element of $\mathcal{S}_1$ between s and $2s$, it follows that $-s \in \Gamma(2s)$, and so, by symmetry of $\Gamma(2s)$, $s \in \Gamma(2s)$. But then there is no element of $\mathcal{S}_1$ between $2s$ and $3s$, and, since $2s \notin \mathcal{S}_1$, it follows that there is no element of $\mathcal{S}_1$ between s and $3s$. Then

$2s \in \Gamma(s)$, and so, by symmetry, $-2s \in \Gamma(s)$. But this implies that there is no element of $\mathcal{S}_1$ between $-s$ and s. But $0 \in \mathcal{S}_1$ and $-s < 0 < s$. This contradiction proves that $S = \{0\}$.

Lemma 19.6. If $S = \{0\}$, then each $\Gamma(x)$ is a group.

Proof. Suppose $S = \{0\}$.

Part a). We show that each $\Gamma(x)$ is symmetric. Suppose $t \in \Gamma(x)$.

Case (i). $t > 0$. Suppose $-t \in \Gamma(x)$. Then there is a g in $\mathcal{S}_1$ such that $x - t < g < x$. Let τ be $x - g$. Then $\tau > 0$, and since $S = \{0\}$ there is an h in $\mathcal{S}_1$ such that $\tau < h < 2\tau$. We observe that $\tau < t$. Then

$$\begin{aligned} x = \tau + g < h + g &< 2\tau + g \\ &= \tau + \tau + g \\ &= \tau + x \\ &< t + x = x + t\ . \end{aligned}$$

But $h + g \in \mathcal{S}_1$, and so $t \in \Gamma(x)$. This contradicts our assumption that $t \in \Gamma(x)$. Thus, in Case (i), $-t \in \Gamma(x)$.

Case (ii). $t < 0$. Suppose $-t \notin \Gamma(x)$. Then there is a g in $\mathcal{S}_1$ such that $x < g < x - t$. Let τ be $g - x$. Then $\tau > 0$, and since $S = \{0\}$ there is an h in $\mathcal{S}_1$ with $\tau < h < 2\tau$. We observe that $\tau < -t$. Then

$$x + t < x - \tau = g - 2\tau < g - h < g - \tau = x\ .$$

But $g - h \in \mathcal{S}_1$, and so $t \notin \Gamma(x)$, contradicting our assumption that $t \in \Gamma(x)$. Thus, in Case (ii), $-t \in \Gamma(x)$.

This concludes the proof of Part (a).

Part (b). By (a), and the convexity of each $\Gamma(x)$, we will have proved the lemma as soon as we have proved that if $t \in \Gamma(x)$ and $t > 0$ then $2t \in \Gamma(x)$.

Suppose $t \in \Gamma(x)$ and $t > 0$ and $2t \notin \Gamma(x)$. Then there is a g in $\mathcal{S}_1$ such that $x < g < x + 2t$. Then, since $t \in \Gamma(x)$, $x + t \leq g < x + 2t$. We distinguish two cases.

Case (i). $x + t = g$. Select h' in $\mathcal{S}_1$ with $t/4 < h' < t/2$. Then one may easily check that $x < g - h' < x + t$, and since $g - h' \in \mathcal{S}_1$, $t \notin \Gamma(x)$, contrary to assumption.

Case (ii). $x + t < g < x + 2t$. Let τ be $g - (x+t)$. Then $\tau > 0$. Select h in $\mathcal{S}_1$ with $\tau < h < 2\tau$. We observe that $\tau < t$. Then $x < x + t - \tau = g - 2\tau < g - h < g - \tau = x + t$. But $g - h \in \mathcal{S}_1$, and so $t \notin \Gamma(x)$, contrary to assumption.

Thus if $t > 0$ and $t \in \Gamma(x)$, then $2t \in \Gamma(x)$. This completes the proof of the lemma.

19.7. After Lemma 5.13 we mentioned the convex valuation v^Z on an ordered field $\mathcal{S}$. We observed that, for x, y in $\mathcal{S}^*$, $v^Z(x) = v^Z(y)$ if and only if there are positive integers m, n such that $|x| \leq m|y|$ and $|y| \leq n|x|$. It is clear that by the latter property we may define a convex group-valuation on an arbitrary ordered group. We denote the resulting group-valuation by "v^Z" also, without risk of confusion.

Lemma 19.8. If $x \in \mathcal{S}_1^*$, then $x \in S$ if and only if $v^Z(x) \notin v^Z[\mathcal{S}_2^*]$.

Proof. (It suffices to prove the above statement for x positive, since S is symmetric, and $v^Z(x) = v^Z(-x)$).

a) Suppose $x > 0$ and $v^Z(x) \notin v^Z[\mathcal{S}_2^*]$. Suppose $x \notin S$. Then there is a y in $\mathcal{S}_2$ with $x < y < 2x$. Then by convexity $v^Z(2x) \leq v^Z(y) \leq v^Z(x)$. Since $v^Z(2x) = v^Z(x)$, it follows that $v^Z(x) = v^Z(y) \in v^Z[\mathcal{S}_2^*]$, which contradicts our assumption. Therefore if $v^Z(x) \notin v^Z[\mathcal{S}_2^*]$ then $x \in S$.

b) Suppose $x > 0$ and $v^Z(x) \in v^Z(\mathcal{S}_2^*]$. Then there is a positive y in $\mathcal{S}_2$, and positive integers m, n such that $x \leq my$ and $y \leq mx$. Then $x \leq my \leq mnx$, and $my \in \mathcal{S}_2$. We distinguish two cases.

Case (i). For some positive integer k, $my = kx$. Then $x \in \mathcal{S}_2$, by divisibility. Then $3x/2 \in \mathcal{S}_2$, and $x < 3x/2 < 2x$, so $x \notin S$.

Case (ii). For all positive integers k, $my \neq kx$. Then there is a positive integer ℓ such that $\ell x < my < (\ell+1)x$. Then

$$x < m/\ell \cdot y < \left(\frac{\ell + 1}{\ell}\right) \cdot x \leq 2x .$$

Since $m/\ell \cdot y \in \mathcal{S}_2$, it follows that $x \notin S$.

Therefore if $v^Z(x) \in v^Z[\mathcal{S}_2^*]$, then $x \notin S$. This completes the proof.

Corollary 19.9. $S = \{0\}$ if and only if $v^Z[\mathcal{S}_1^*] = v^Z[\mathcal{S}_2^*]$.

Proof. Trivial, by 19.8.

Lemma 19.10. Suppose $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair of ordered fields.

a) If $S \neq \{0\}$, then S is cofinal and coinitial in $\mathcal{S}_1$.

b) If $s \in S$ and $n \in Z$, and $x^n = s$, then $x \in S$.

c) There is no s in S with $1 < s < 2$.

Proof. a) Suppose $s \in S$ and $s \neq 0$. Then $v^Z(s) \notin v^Z[\mathcal{S}_2^*]$, by 19.8. Without loss of generality, $s > 0$. We have $v^Z(s) \neq 0$. Let x be an arbitrary positive element of $\mathcal{S}_1$. There are two cases.

Case (i). $x \notin S$. Then by 19.8, $v^Z(x) \in v^Z[\mathcal{S}_2^*]$. It follows that $v^Z(sx) \notin v^Z[\mathcal{S}_2^*]$, and $v^Z(s^{-1}x) \notin v^Z[\mathcal{S}_2^*]$. We have, therefore, by 19.8, that $sx \in S$ and $s^{-1}x \in S$. We have either $0 < sx < x < s^{-1}x$, or $0 < s^{-1}x < x < sx$.

Case (ii). $x \in S$. Then by 19.8, $v^Z(x) \notin v^Z[\mathcal{S}_2^*]$, and so $v^Z(x/2) \notin v^Z[\mathcal{S}_2^*]$, and $v^Z(2x) \notin v^Z[\mathcal{S}_2^*]$. Again by 19.8, $x/2 \in S$, and $2x \in S$. We have $0 < x/2 < x < 2x$.

This proves (a).

b) Suppose $s \in S$ and $x^n = s$. By 19.8, $v^Z(s) \notin v^Z[\mathcal{S}_2^*]$. If $v^Z(x) \in v^Z[\mathcal{S}_2^*]$, then $nv^Z(x) \in v^Z[\mathcal{S}_2^*]$, and so $v^Z(x^n) \in v^Z[\mathcal{S}_2^*]$. Therefore $v^Z(x) \notin v^Z[\mathcal{S}_2^*]$, and, by 19.8, $x \in S$.

c) Suppose $s \in S$ and $1 < s < 2$. Then $v^Z(2) \leq v^Z(s) \leq v^Z(1)$. Therefore $v^Z(s) = 0 \in v^Z[\mathcal{S}_2^*]$. Therefore, by 19.8, $s \notin S$. Therefore there is no s in S with $1 < s < 2$.

This completes the proof.

Definition 19.11. (Suppose $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair of ordered fields). We define A as $\{x | x > 1 \wedge (\forall s \in S) \neg B(1,s,x)\}$.

Lemma 19.12. If $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair of real-closed fields, then A is quasi-archimedean.

Proof. Suppose $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair of real-closed fields. Then 19.10(c) shows that $2 \in A$. All elements of A are positive, by definition. Thus, to show that A is quasi-archimedean, we have only to show that A is closed under multiplication.

Suppose $x_1 \leq x_2$, and $x_1 x_2 \notin A$. Then there is an s in S with $1 < s < x_1 x_2 \leq x_2^2$. Let $\sqrt{s}$ be the positive square root of s. Then, by 19.10 (b), $\sqrt{s} \in S$. But $1 < \sqrt{s} < x_2$, and so $x_2 \notin A$.

It follows that A is closed under multiplication, and we are done.

19.13. For the remainder of this section, $(\mathcal{S}_1, \mathcal{S}_2)$ is to be a pair of real-closed fields. By 19.12 we get a convex valuation v^A on $\mathcal{S}_1$. Let G_1 be the D-group $v^A[\mathcal{S}_1^*]$, and G_2 the D-subgroup $v^A[\mathcal{S}_2^*]$, and G_2 the D-subgroup $v^A[\mathcal{S}_2^*]$. Let H_1 be the D-group $v^Z[\mathcal{S}_1^*]$ and H_2 the D-subgroup $v^Z[\mathcal{S}_2^*]$. Let H_3 be the unique maximal subgroup of H_2 which is convex in H_1.

Let A^{-1} be $\{x | x^{-1} \in A\}$, and let $\hat{A}$ be $A^{-1} \cup \{1\} \cup A$. Using 19.10(b) with $n = -1$, we see easily that $\hat{A} \cap S = \emptyset$, and $\hat{A}$ is a convex set of positive elements.

Lemma 19.14. If x is positive, then $x \in \hat{A}$ if and only if $v^Z(x) \in H_3$.

Proof. a) Suppose $x \in \hat{A}$, but $v^Z(x) \notin H_3$. Then there is a γ in $H_1 \sim H_2$ such that $0 < \gamma < |v^Z(x)|$. Let s be a positive element with $v^Z(s) = \gamma$. Then, by convexity, since $v^Z(1) < v^Z(s) < |v^Z(x)|$, it follows that either $x < s < 1$ or $x^{-1} < s < 1$. Now $s \in S$, since $v^Z(s) \notin v^Z[\mathcal{S}_2^*]$. It follows that $x \notin \hat{A}$, a contradiction. Therefore if $x \in \hat{A}$ then $v^Z(x) \in H_3$.

b) Suppose $x > 0$ and $v^Z(x) \in H_3$, and suppose $x \notin \hat{A}$. Then there is an s in S with either $x < s < 1$ or $1 < s < x$. In either case $|v^Z(s)| \leq |v^Z(x)|$, whence $v^Z(s) \in H_3 \subset H_2$. But since $s \in S$, $v^Z(s) \notin H_2$. Therefore, if $x > 0$ and $v^Z(x) \in H_3$, then $x \in \hat{A}$. This completes the proof.

Lemma 19.15. v^A (the valuation-ring of A) is equal to $\{x | (\exists \gamma \in H_3)(v^Z(x) \geq \gamma)\} \cup \{0\}$.

Proof. According to 5.12, v^A is equal to $\{x | (\exists y \in A)(|x| \leq y)\}$.

a) If $y \in A$ and $|x| \leq y$, then either $x = 0$ or $v^Z(x) \geq v^Z(y) \in H_3$.

b) If $\gamma \in H_3$ and $v^Z(x) \geq \gamma$, then $v^Z(x) \geq v^Z(y)$ for some positive y in $\hat{A}$. But then for some integer n, $|x| \leq 2^n \cdot y$. If $y \in A^{-1}$, then $|x| \leq 2^n$, and so $|x| \leq y_1$ for some y_1 in A. If $y \in A \cup \{1\}$, then $2^n \cdot y \in A$, and so $|x| \leq y_2$ for some y_2 in A. Thus in either case, $|x|$ is bounded above by an element of A. This completes the proof.

Corollary 19.16. If $x \in \mathcal{S}_1^*$, then $v^A(x) = 0$ if and only if $v^Z(x) \in H_3$.

Proof. Trivial by 19.15, and the convexity of H_3.

Lemma 19.17. $G_1 \sim G_2$ is coinitial in G_1.

Proof. G_1 and G_2 are D-groups. If $\delta \in G_1 \sim G_2$, then $\delta/2 \in G_1 \sim G_2$, so that if $0 < \delta$, then $0 < \delta/2 < \delta$.

Suppose $x \in \mathcal{S}_2^*$ and $v^A(x) > 0$. Then, using 19.15, $v^Z(x) > \gamma$ for all γ in H_3. By the definition of H_3 as the maximal subgroup of H_2 that is convex in H_1, there is a γ_1 in $H_1 \sim H_2$ such that $v^Z(x) > \gamma_1 > 0$. But, by 19.8, $\gamma_1 = v^Z(s)$ for some s in S. By convexity, $|x| < |s| < 1$. But then $v^A(x) \geq v^A(s) \geq 0$. If $v^A(s) = 0$, then by 19.16, $v^Z(s) \in H_3 \subset H_2$, contrary to 19.8. Therefore, $v^A(s) > 0$. If $v^A(x) = v^A(s)$ then $v^A(x/s) = 0$, so by 19.16 $v^Z(x/s) \in H_3$. But then $v^Z(x) - v^Z(s) \in H_3$, and since $v^Z(x) \in H_2$, it follows that $v^Z(s) \in H_2$, contrary to 19.8. Therefore $v^A(x) > v^A(s) > 0$. But clearly $S \cap \mathcal{S}_2 = \emptyset$, and so $v^A(s) \in G_1 \sim G_2$.

This proves that if $\delta \in G_2$ and $0 < \delta$, there is an ε in $G_1 \sim G_2$ such that $0 < \varepsilon < \delta$. This concludes the proof.

Section 20. The Elementary Theory of Pairs of D-Groups

In this section we give a very brief indication of what we mean by the elementary theory of pairs of D-groups. We work with an applied first-order predicate logic $\mathcal{L}'$ with identity. $\mathcal{L}'$ has the usual symbols $=, \wedge, \vee, \neg, \rightarrow, \longleftrightarrow, \forall, \exists$, and individual variables $x_0, x_1, \ldots, y_0, y_1, \ldots, z_0, z_1, \ldots$. The above symbols get the usual interpretation. In addition $\mathcal{L}'$ has an individual constant 0, a binary operation-symbol +, a binary relation-symbol <, and a unary predicate-symbol H. A model of $\mathcal{L}'$ is a 5-tuple $\langle G, H, +, 0, < \rangle$, where G is a non-empty set, H is a subset of G, + is a binary relation from $G \times G$ into G, 0 is an element of G, and < is a binary relation on G. As in 17, we use the same symbols for symbols of $\mathcal{L}'$ and for the interpretation of those symbols in a model. Also, we identify a model $\langle G, H, +, 0, < \rangle$ with its domain G.

It is clear what we intend when we say that a model of $\mathcal{L}'$ is a pair (G,H) of D-groups. It is equally clear that the class of pairs of D-groups has a recursive axiomatization in $\mathcal{L}'$. Let $\mathcal{B}$ be a fixed recursive set of axioms in $\mathcal{L}'$ for pairs of D-groups, and let U be the set of logical consequences of $\mathcal{B}$. Then U is the elementary theory of pairs of D-groups.

Remark. The theory of a single D-group is slightly complicated by the fact that there are trivial, one-element D-groups. All non-trivial D-groups are elementarily equivalent.

Let us assume the notion of relativization to H has been defined, just as relativization to L was defined in 17. The following lemma is simply a restatement of A. Robinson's theorem on the model-completeness of the theory of non-trivial D-groups.

Theorem 20.1. Let $\varphi(x_o,\ldots,x_k)$ be a formula of $\mathcal{L}'$ in which H does not occur, and in which $x_o, \ldots, x_k$ are the only free variables. Then φ^+ is in U, where φ^+ is

$$(\forall x_o) \cdots (\forall x_k)[((\exists y_o)(Hy_o \wedge y_o \neq 0) \wedge Hx_o \wedge \cdots \wedge Hx_k \wedge \varphi(x_o,\ldots,x_k)) \rightarrow \varphi^H(x_o,\ldots,x_k)] .$$

Section 21. There are $2^{\aleph_0}$ Elementary Types of Pairs of Real-Closed Fields

21.1. In 19.11 we defined, for a pair $(\mathcal{S}_1, \mathcal{S}_2)$ of ordered fields, the sets S and A, and the valuation v^A. Suppose now that $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair (K,L), where $\langle K, L, +, \cdot, 0, 1, < \rangle$ is a model of $\mathcal{A}$, the axioms for a pair of real-closed fields. It is completely clear that S and A are definable by formulae of $\mathcal{L}$. Since A is definable, we can interpret within T the theory of the pair of D-groups $(v^A[K^*], \ v^A[L^*])$.

We want to find out which pairs (G,H) of D-groups can occur as the pair $(v^A[K^*], \ v^A[L^*])$, for some pair (K,L) of real-closed fields. In Lemma 19.17 we showed that $v^A[K^*] \sim v^A[L^*]$ is coinitial in $v^A[K^*]$. Thus a necessary condition, for (G,H) to occur as a pair $(v^A[K^*], v^A[L^*])$, is that $G \sim H$ is coinitial in G. The following lemma shows that this condition is also sufficient.

Lemma 21.2. Suppose (G,H) is a pair of D-groups with $G \sim H$ coinitial in G. Let $\mathbb{R}$ be the reals. Let $\mathcal{S}_1$ be the real-closed field $\mathbb{R}((t^G))$, and $\mathcal{S}_2$ the real-closed field $\mathbb{R}((t^H))$. Then, for the pair $(\mathcal{S}_1, \mathcal{S}_2)$ we have $(v^A[\mathcal{S}_1^*], v^A[\mathcal{S}_2^*]) \cong (G,H)$.

Proof. Let G, H, $\mathcal{S}_1$, $\mathcal{S}_2$ be as in the statement of the lemma. It is clear that $(v^Z[\mathcal{S}_1^*], v^Z[\mathcal{S}_2^*]) \cong (G,H)$. It follows that $v^Z[\mathcal{S}_1^*] \sim v^Z[\mathcal{S}_2^*]$ is coinitial in $v^Z[\mathcal{S}_1^*]$. Therefore $\{0\}$ is the unique maximal subgroup of $v^Z[\mathcal{S}_2^*]$ that is convex in $v^Z[\mathcal{S}_1^*]$. Then, by 19.15, v^A and v^Z have the same valuation-ring. Therefore

$$(v^A(\mathcal{S}_1^*],\ v^A[\mathcal{S}_2^*]) \cong (v^Z[\mathcal{S}_1^*],\ v^Z[\mathcal{S}_2^*]) \cong (G,H)\ .$$

This proves the result.

It follows that we can interpret within T the theory of pairs (G,H) of D-groups, subject to the condition that $G \sim H$ is coinitial in G. Thus there are at least as many elementary types of pairs of real-closed fields as there elementary types of pairs (G,H) of D-groups, subject to the condition that $G \sim H$ is coinitial in G.

21.3. Let Θ be the sentence (of $\mathcal{L}'$)

$$(\forall x_o)(\exists y_o)[x_o < 0 \vee x_o = 0 \vee (0 < y_o \wedge y_o < x_o \wedge \neg Hy_o)]\ .$$

Θ holds in a model $\langle G, H, +, 0, < \rangle$ if and only $G \sim H$ is coinitial in G. Let $\mathcal{B}_1$ be $\mathcal{B} \cup \{\Theta\}$, and let U_1 be the set of logical consequences of $\mathcal{B}_1$. Then U_1 is the theory of pairs (G,H) of D-groups, subject to the condition that $G \sim H$ be coinitial in G. Our immediate aim is to show that U_1 has $2^{\aleph_o}$ complete consistent extensions. We will do this by interpreting within U_1 the theory of an arbitrary linear order with first element.

In 19.11 we defined the set S for a pair $(\mathcal{S}_1, \mathcal{S}_2)$ of D-groups. Suppose that $(\mathcal{S}_1, \mathcal{S}_2)$ is a pair (G,H), where $\langle G, H, +, 0, < \rangle$ is a model of $\mathcal{B}$. It is clear that S is definable by a formula of $\mathcal{L}'$. We recall that $0 \in S$.

Let (G,H) be an arbitrary model of $\mathcal{B}$. We are going to give informal definitions, both of an equivalence relation E on the non-negative elements of S, and of a linear order W on the set $Cl(E)$ of equivalence classes of E. It will be clear that E is definable in the logic $\mathcal{L}'$, and equally clear that we may interpret in U the theory of the ordered set $\langle Cl(E), W\rangle$.

Definition of E. For s_1, s_2 in S, with $s_1 \geq 0$ and $s_2 \geq 0$,

$$E(s_1,s_2) =_{def} (\forall x)[B(s_1,x,s_2) \rightarrow x \in S] .$$

It is clear that E is an equivalence relation on the non-negative elements of S.

Definition of W. If X_1, X_2 are equivalence classes of E, then

$$W(X_1,X_2) =_{def} (\forall s_1 \in X_1)(\forall s_2 \in X_2)[s_1 < s_2] .$$

One may easily check that W is a linear order on $Cl(E)$, the set of equivalence classes of E. Let X_o be the equivalence class of 0. Then X_o is the least element of $\langle Cl(E), W\rangle$.

We want to find out which ordered sets $\langle X, <_X\rangle$ are isomorphic to an ordered set $\langle Cl(E), W\rangle$ for a pair (G,H) such that $G \sim H$ is coinitial in G. We know from the previous paragraph that $Cl(E)$ has a least element under W. Thus if $\langle X, <_X\rangle$ is isomorphic to a pair $\langle Cl(E), W\rangle$, then X has a least element under $<_X$.

The converse is true, as we now show. Suppose $\langle X, <_X\rangle$ is an ordered set ∇ with least element x_o. Let ∇' be the converse ordering, i.e., $\langle X, >_X\rangle$. Let J be the set $\{0, 1\}$, ordered by $0 <_J 1$. Let Λ_o be the lexicographic product $J \times \nabla' \times J$. We imbed ∇' in Λ_o by the map $x \rightsquigarrow \langle 1, x, 0\rangle$, and we identify ∇' with its image in Λ_o. Let Λ_1 be $\Lambda_o \sim \{\langle 1, x_o, 1\rangle\}$. Let Λ_2 be $\Lambda_1 \sim (\Lambda_1 \cap \nabla')$. Then (Λ_1, Λ_2) is a pair of ordered sets, with the following three properties, which may easily be checked:

i) For each ℓ in Λ_1 there is an ℓ' in $\Lambda_1 \sim \Lambda_2$ with with $\ell \leq_{\Lambda_1} \ell'$;

ii) If $\ell <_{\Lambda_1} \ell'$ and ℓ and ℓ' are in $\Lambda_1 \sim \Lambda_2$, there is an ℓ'' in Λ_2 with $\ell <_{\Lambda_1} \ell'' <_{\Lambda_1} \ell'$;

iii) $\Lambda_1 \sim \Lambda_2 \cong \nabla'$.

Let G be the Hahn group $\Gamma_{\ell \in \Lambda_1} \mathbb{R}_\ell$, where $\mathbb{R}_\ell = \mathbb{R}$ for all ℓ. Let H be the Hahn group $\Gamma_{\ell \in \Lambda_2} \mathbb{R}_\ell$. Then G and H are D-groups, and H is a subgroup of G. Because of (i) above, $G \sim H$ is coinitial in G.

Let $v: G^* \to \Lambda_1$ be the canonical valuation. One may easily check that if $g \in G^*$ then $g \in S$ if and only if $v(g) \notin \Lambda_2$. If s_1, $s_2 \in S$ and $s_1 > 0$, $s_2 > 0$, then, by (ii) above, $E(s_1,s_2)$ if and only if $v(s_1) = v(s_2)$. Also by (ii), if $s \in S$ and $s > 0$ then $E(0,s)$ if and only if $v(s) = \langle 1, x_o, 0\rangle$. It follows easily that $\langle Cl(E)), W\rangle$ is isomorphic to the converse ordering of $\Lambda_1 \sim \Lambda_2$, i.e., by (iii) to the converse ordering of ∇ , i.e., to ∇.

This completes the proof of the converse.

We have thus shown that an ordered set ∇ is isomorphic to a system $\langle Cl(E), W\rangle$, for a pair (G,H) such that $G \sim H$ is coinitial in G, if and only if ∇ has a least element. Thus we can interpret in U_1 the theory O_1 of an arbitrary linear order with first element. Therefore U_1 has at least as many complete consistent extensions as O_1 has.

It is known that O_1 has $2^{\aleph_0}$ complete consistent extensions. We sketch a proof of this result. Let $\langle\alpha_n\rangle_{n<\omega}$ be a sequence of 0's and 1's. To $\langle\alpha_n\rangle_{n<\omega}$ we assign an order-type $\zeta(\langle\alpha_n\rangle_{n<\omega})$ by the rule:

$$\zeta(\langle\alpha_n\rangle_{n<\omega}) = 1 + \sum_{n<\omega}\beta_n ,$$

where

$$\beta_n = \omega + 1 + \omega^* \quad \text{if} \quad \alpha_n = 0 ,$$

and

$$\beta_n = 2 + \omega^* \quad \text{if} \quad \alpha_n = 1 .$$

Then $\zeta(\langle\alpha_n\rangle_{n<\omega})$ has a first element. Suppose $\langle\alpha_n\rangle_{n<\omega}$ and $\langle\alpha'_n\rangle_{n<\omega}$ are distinct sequences. Let $\zeta(\langle\alpha_n\rangle_{n<\omega})$ be $1 + \sum_{n<\omega}\beta_n$, and let $\zeta(\langle\alpha'_n\rangle_{n<\omega})$ be $1 + \sum_{n<\omega}\beta'_n$, where β_n, β'_n are determined by the rule above. Let k be the least n for which $\alpha_n \neq \alpha'_n$, and suppose without loss of generality that $\alpha_k = 0$, $\alpha'_k = 1$. We now describe a first-order property distinguishing $\zeta(\langle\alpha_n\rangle_{n<\omega})$ from $\zeta(\langle\alpha'_n\rangle_{n<\omega})$. In $\zeta(\langle\alpha_n\rangle_{n<\omega})$, the k^{th}, among the elements that have no immediate successor, has no immediate predecessor. In $\zeta(\langle\alpha'_n\rangle_{n<\omega})$, the k^{th} among the elements that have no immediate successor, has an immediate predecessor.

It follows that there are at least $2^{\aleph_0}$ elementary types of models of O_1. But, in general, for a countable logic, there are at most $2^{\aleph_0}$ elementary types of models of a given set of sentences. Thus there are exactly $2^{\aleph_0}$ complete, consistent extensions of O_1.

By the preceding discussion we have now established:

Theorem 21.4. a) There are $2^{\aleph_0}$ elementary types of pairs (G,H) of D-groups satisfying the condition that $G \sim H$ is coinitial in G.

b) There are $2^{\aleph_0}$ elementary types of pairs (K,L) of real-closed fields.

Remark. Closer inspection of the preceding arguments shows that there are $2^{\aleph_0}$ elementary types of pairs (G,H) of D-groups satisfying the condition that $G \sim H$ is coinitial in G, and $S \neq \{0\}$.

Remark. By 3.7 we see that, for pairs (G,H) of D-groups, $G \sim H$ is coinitial in G if and only if $G \sim H$ is dense in G. Thus there are $2^{\aleph_0}$ elementary types of pairs (G,H) of D-groups subject to the condition that $G \sim H$ is dense in G.

Section 22. The Case $S = \{0\}$

In this section we discuss pairs (K,L) of real-closed fields subject to the condition $S = \{0\}$. It is clear that the condition $S = \{0\}$ is elementary. We will say that L is weakly dense in K if $S = \{0\}$. It is clear that if L is weakly dense in K, then L is cofinal in K. It is equally clear that if L is dense in K, then L is weakly dense in K.

Lemma 22.1. There exist pairs (K,L) of real-closed fields such that L is weakly dense in K, but L is not dense in K.

Proof. Let $\tilde{Q}$ be the field of real algebraic numbers. Let G be a non-trivial D-group. Let K be $\mathbb{R}((t^G))$, and let L be $\tilde{Q}((t^G))$. Since $\mathbb{R}$ and $\tilde{Q}$ are archimedean it follows that $v^Z[K] = G = v^Z[L]$. By 19.9, $S = \{0\}$ for the pair (K,L).

Now let γ be a positive element of G, and let μ be a transcendental real number. Let x be the element $\mu \cdot t^{\gamma}$. It is simple to check that for any ℓ in L, $|x - \ell| \geq t^{2\gamma}$. Thus L is not dense in K, and the lemma is proved.

Remark. Later we show, by a different technique, that there are at least $\aleph_0$ elementary types of pairs (K,L) of real-closed fields subject to the condition that $S = \{0\}$ and L is not dense in K.

By inspection of the proof of 22.1, we see that for the pair (K,L), constructed in that proof, L is closed in K. We can prove 22.1 by a different construction, as in the following lemma.

Lemma 22.2. There exist pairs (K,L) of real-closed fields such that L is weakly dense in K, L is not dense in K, and L is not closed in K.

Proof. Let K be $\mathbb{R}((t^Q))$, and let L be the real-closure in K of $\tilde{Q}(t)$. We see easily that the residue-class field of L, with respect to v^Z, is $\tilde{Q}$, so L is not dense in K.

Clearly L is countable, and K has cardinality $2^{\aleph_0}$. Since L contains $t^{m/n}$ for all integers m, n with $n \neq 0$, it is clear that $v^Z[K] = v^Z[L] = Q$. Thus, by 19.9, $S = \{0\}$ for the pair (K,L).

Let K_o be $\tilde{Q}((t^Z))$. K_o is a subfield of K. (K_o is not real-closed.) Clearly K_o is of cardinality $2^{\aleph_0}$, so K_o is not a subset of L. We now show that K_o is a subfield of $\bar{L}$, the closure of L in K.

Let x be an arbitrary element of K_o. Then for $n \in Z$ there are c_n in $\tilde{Q}$, and $N \in Z$ such that $c_n = 0$ if $n < N$, such that

$$x = \sum_{n \in Z} c_n \cdot t^n .$$

Since Z is cofinal in Q, it is completely clear that x may be approximated arbitrarily closely in K by its partial sums, which are elements of L. Thus K_o is a subfield of $\bar{L}$.

It follows that $L \neq \bar{L}$, i.e., L is not closed in K.

22.3. Suppose M is a model (K,L) of $\mathcal{A}$, for which $S = \{0\}$. It is obvious that for the model $\operatorname{clos}(M)$ we have $S = \{0\}$ also.

Conversely, suppose M is a model (K,L) of $\mathcal{A}$, such that in the model $\operatorname{clos}(M)$ we have $S = \{0\}$. We claim $S = \{0\}$ for (K,L).

For suppose $x \in K$ and $x > 0$. Then there is a λ in $\bar{L}$ such that $x < \lambda < 2x$. Now there is an ℓ in L such that $|\lambda - \ell| < \min(|x - \lambda|, |2x - \lambda|)$. Therefore $x < \ell < 2x$, and so $S = \{0\}$ for the pair (K,L).

As in 17.3, let $\mathfrak{M}$ be the class of pairs (K,L) of real-closed fields. Let $\mathfrak{M}_S$ be the subclass of $\mathfrak{M}$ consisting of those pairs for which $S = \{0\}$. Then the preceding paragraphs show that the map <u>clos</u> maps $\mathfrak{M}_S$ into $\mathfrak{M}_S$, and $\mathfrak{M} \sim \mathfrak{M}_S$ into $\mathfrak{M} \sim \mathfrak{M}_S$.

<u>22.4</u>. Our final result will be that $\mathfrak{M}_S$ contains at least $\aleph_0$ elementary types. In order to motivate our construction we need certain preliminaries. Recall the sets $\Gamma(x)$ introduced in Section 19.

<u>Lemma 22.5</u>. Suppose (K,L) is a pair of real-closed fields, for which $S = \{0\}$. Let x be any element of K and define

$$\Gamma_1(x) = \{y \in K \,|\, (\forall \ell \in L)(v^Z(y) > v^Z(x-\ell))\} ,$$

where we adopt the convention that $v^Z(0) = \infty > v^Z(k)$ for all k in K. Then

$$\Gamma_1(x) = \Gamma(x) .$$

<u>Proof</u>. Assume the hypotheses of the lemma.

Suppose first $y \in \Gamma(x)$. Then there is an ℓ in L such that $B(x, \ell, x + y)$. Then, by the convexity of v^Z, we get $v^Z(y) \leq v^Z(x-\ell)$, so $y \in \Gamma_1(x)$. Therefore $\Gamma_1(x) \subseteq \Gamma(x)$. (We did not use $S = \{0\}$ for this).

Suppose now $y \notin \Gamma_1(x)$. Then there is an ℓ in L and a positive integer n such that $|x - \ell| \leq n \cdot |y|$. Since $S = \{0\}$, $\Gamma(x)$ is symmetric, and so $ny \notin \Gamma(x)$. But, again since $S = \{0\}$, $\Gamma(x)$ is a group and so $y \notin \Gamma(x)$. Therefore $\Gamma(x) \subseteq \Gamma_1(x)$.

It follows that $\Gamma(x) = \Gamma_1(x)$.

22.6. From 22.5 we see that if $(K,L) \in \mathcal{M}_S$ and $x \in K$ then $\Gamma(x)$ is of the form

$$\{y \mid v^Z(y) > B\} ,$$

where B is a subset of the group $v^Z[K]$.

For an arbitrary subset B of $v^Z[K]$ we define $\theta(B)$ as $\{y \mid v^Z(y) > B\}$. It is obvious that $\theta(B)$ is a convex subgroup of K. Clearly $1 \in \theta(B)$ if and only if $0 > B$. (When $0 > B$ we say B is negative).

Suppose B is negative. We define $\check{B}$ as the set of those γ in $v^Z[K]$ such that $\gamma > B$ and $-\gamma > B$. Since B is negative, $0 \in \check{B}$. It is clear that $\check{B}$ is convex and symmetric.

Suppose $\check{B}$ is a group. Suppose $y_1, y_2 \in \theta(B)$. Then $v^Z(y_1) > B$, and $v^Z(y_2) > B$. We have $v^Z(y_1y_2) = v^Z(y_1) + v^Z(y_2)$. If $y_1y_2 \notin \theta(B)$, there is a b in B such that $v^Z(y_1) + v^Z(y_2) \leq b$. Then without loss of generality $v^Z(y_1) \leq b/2$. Since $y_1 \in \theta(B)$ it follows that $b/2 > B$. Therefore $b/2 > b$, so $b < 0$. It follows that $b/2 \in \check{B}$, and since $\check{B}$ is a group, $b \in \check{B}$. But then $b > b$, a contradiction. It follows that $y_1y_2 \in \theta(B)$. We have thus shown that if $\check{B}$ is a group then $\theta(B)$ is a convex subring of K.

Suppose conversely that $\theta(B)$ is a convex subring of K. Suppose $\gamma_1, \gamma_2 \in \check{B}$. Select y_1, y_2 in K such that $v^Z(y_1) = \gamma_1$ and $v^Z(y_2) = \gamma_2$. Then, since $\gamma_1 > B$ and $\gamma_2 > B$, it follows that $y_1 \in \theta(B)$ and $y_2 \in \theta(B)$. But then $y_1 y_2 \in \theta(B)$, and so $\gamma_1 + \gamma_2 = v^Z(y_1) + v^Z(y_2) > B$. A similar argument shows that $-(\gamma_1 + \gamma_2) > B$. Therefore, if $\gamma_1, \gamma_2 \in \check{B}$ then $\gamma_1 + \gamma_2 \in B$, and so, by the symmetry of $\check{B}$, $\check{B}$ is a group. We have thus shown that if $\theta(B)$ is a convex subring, then $\check{B}$ is a group.

Thus if B is negative, a necessary and sufficient condition for $\theta(B)$ to be a convex subring of K is that $\check{B}$ should be a subgroup of $v^Z[L]$.

We notice that we may define $\theta(B)$ in terms of $\check{B}$ by:

$$\theta(B) = \{y | (\exists \gamma \in \check{B})(v^Z(y) \geq \gamma) \quad .$$

We leave the verification of this as a simple exercise for the reader. From the preceding considerations we deduce the following lemma.

Lemma 22.7. Suppose $(K,L) \in \mathfrak{M}_S$, and $x \in K$. Then a necessary and sufficient condition that $\Gamma(x)$ should be a subring of K is that there is a convex subgroup H of $v^Z[K]$ such that

$$\Gamma(x) = \{y \; (\exists \gamma \in H)(v^Z(y) \geq \gamma)\} \quad .$$

Corollary 22.8. Suppose $(K,L) \in \mathfrak{M}_S$, and $v^Z[K]$ is archimedean. Then if $x \in K$ and $\Gamma(x)$ is a subring of K, we have

$$\Gamma(x) = \{y | v^Z(y) \geq 0\} \quad .$$

Proof. Assume the hypotheses of the corollary. The only convex subgroups of $v^Z[K]$ are $\{0\}$ and $v^Z[K]$. We observe that we cannot have $\Gamma(x) = K$, and our result follows by 22.7.

Lemma 22.9. There are (K,L) in $\mathfrak{M}_S$, with L not dense in K, such that for no x in K is $\Gamma(x)$ a subring of K.

Proof. As in 22.1, let K be $\mathbb{R}((t^Q))$, and let L be $\tilde{Q}((t^Q))$. Since Q is archimedean we may identify v^Z with the canonical valuation v of K onto Q. It is quite clear that if $x \in K$, then either $\Gamma(x) = \{0\}$ or there is an r in Q such that $\Gamma(x) = \{y \mid v^Z(y) > r\}$. For suppose $x \in K - L$. Then x is of the form $\sum_{\gamma\in\Delta} C_\gamma \cdot t^\gamma$, where Δ is a subset of Q well-ordered by the natural order, and the C_γ are in $\mathbb{R}$. Moreover some $C_\gamma \in \mathbb{R} \sim \tilde{Q}$. Let r be the least γ for which this happens. Then one may easily verify that

$$\Gamma(x) = \{y \mid v^Z(y) > r\} .$$

Now we observe that because of the preceding paragraph no $\Gamma(x)$ is of the form $\{y \mid v^Z(y) \geq 0\}$. It follows by 22.8 that no $\Gamma(x)$ is a subring of K.

In contrast, the following simple lemma holds.

Lemma 22.10. Suppose simply that L is cofinal in K, but L is not dense in K. Then there exists an x in K such that $1 \in \Gamma(x)$.

Proof. Suppose L is cofinal in K but not dense in K. Then there is an x in K such that $\Gamma(x) \neq \{0\}$. Suppose $\tau \in \Gamma(x)$ and $\tau \neq 0$. Since L is cofinal in K, there is an ℓ in L such that $\ell\tau > 1$.

Suppose there is an ℓ_1 in L with $B(\ell x \ \ell_1, \ell\tau)$. Then clearly $B(x, \ell_1 \cdot \ell^{-1}, \tau)$, and $\ell_1 \cdot \ell^{-1} \in L$, contradicting the fact that $\tau \in \Gamma(x)$. It follows that $\ell\tau \in \Gamma(\ell x)$, and by convexity $1 \in \Gamma(\ell x)$. This proves the lemma.

<u>22.11</u>. Let (K,L) be in $\mathcal{M}_s$. We define $\mathcal{H}$ as the set $\{\Gamma(x)\}_{x \in K}$. $\mathcal{H}$ is a set of convex subgroups of K. $\mathcal{H}$ is linearly ordered by inclusion, with maximal element $\{0\}$. Let $\mathcal{H}^{\circ}$ be the subset of $\mathcal{H}$ consisting of those $\Gamma(x)$ that are subrings of K. $\mathcal{H}^{\circ}$ is also linearly ordered by inclusion, and does not contain $\{0\}$. We have seen that, even if L is not dense in K, $\mathcal{H}^{\circ}$ may be empty.

Let $\mathcal{A}_s$ be a recursive set of sentences of $\mathcal{L}$, extending $\mathcal{A}$ and axiomatizing $\mathcal{M}_s$. It is clear that such sets exist. We see easily that we may interpret within the theory of $\mathcal{A}_s$ the theory of the ordered set $\mathcal{H}$ and its subset $\mathcal{H}^{\circ}$.

Let $\mathcal{A}_s^+$ be a recursive set of sentences of $\mathcal{L}$, extending $\mathcal{A}_s$, and axiomatizing the class of models (K,L) of $\mathcal{A}_s$ for which $\mathcal{H}^{\circ} \neq \emptyset$. It is clear that such sets exist. Moreover, in any model (K,L) of $\mathcal{A}_s^+$, L is not dense in K.

We see easily that we may interpret within the theory of $\mathcal{A}_s^+$ the theory of the ordered set $\langle \mathcal{H}^{\circ}, \subset \rangle$, where $\subset$ is inclusion. We have been unable to ascertain for which linear orders $\langle X, <_X \rangle$ there is a model (K,L) of $\mathcal{A}_s^+$ such that $\langle \mathcal{H}^{\circ}, \subset \rangle \cong \langle X, <_X \rangle$. (It is consistent with what we know that all linear orders occur in this way.) However, we have the following result.

Lemma 22.12. Suppose $\langle X, <_X\rangle$ is a finite linear order. Then there is a model (K,L) of $\mathcal{a}_s^+$ such that $\langle \mathcal{H}^{\circ}, \subset\rangle \cong \langle X, <_X\rangle$.

Proof. Let $\langle X, <_X\rangle$ be a finite linear order, of cardinal n, say.

Part 1. We construct a D-group G which has exactly n proper convex subgroups.

If $n = 1$, let G be Q. If $n > 1$, let X_1 be the ordered set got by removing from X its greatest element. Let G be $\Gamma_{\ell \in X_1} Q_\ell$, where each Q_ℓ is Q. Then v^Z may be identified with the canonical group-valuation of G onto X_1. It is a simple exercise to show that the proper convex subgroups of G are precisely $\{0\}$ and the groups $\{g \mid v^Z(g) \geq x_1\}$, where $x_1 \in X_1$.

Thus we see that G is a countable D-group with the property that the ordered set consisting of the proper convex subgroups of G, ordered by reverse inclusion, is isomorphic to $\langle X, <_X\rangle$.

Part 2. Let λ be a countable limit ordinal, and let H be a proper convex subgroup of G. We claim there is a well-ordered series $\{g_\mu\}_{\mu<\lambda}$ of elements of G, satisfying (a), (b), (c) below.

(a) $\{g_\mu\}_{\mu<\lambda}$ is increasing in $<_G$, the order on G.

(b) If $\mu < \lambda$ and $h \in H$, then $g_\mu <_G h$.

(c) If $\gamma \in G$ and $\gamma <_G h$ for all h in H, then there is a $\mu < \lambda$ such that $\gamma <_G g_\mu$.

We prove this by a variant of a classical argument of Cantor. First we enumerate the ordinals less than λ, say as $\{\tau_n\}_{n<\omega}$. Next, we enumerate, as $\{\gamma_n\}_{n<\omega}$, those γ such that for all h in H $\gamma <_G h$. We observe that in G $\{\gamma_n\}_{n<\omega}$ is densely ordered without first or last element, since G is a D-group.

We define g_{τ_o} as γ_o. Suppose we have defined g_{τ_n} for all $n < N$, so as to satisfy the following conditions (i), (ii), (iii).

(i) If $n_1, n_2 < N$ and $\tau_{n_1} < \tau_{n_2}$, then $g_{\tau_{n_1}} <_G g_{\tau_{n_2}}$.

(ii) For all $n < N$, and h in H, $g_{\tau_n} <_G h$.

(iii) If $n < N$, and $g_{\tau_n} >_G g_{\tau_m}$ for all $m < n$, then

$$g_{\tau_n} \geq_G \max_{0 \leq j \leq n} \gamma_j.$$

We now define g_{τ_N}. Let E^- be the set of $n < N$ such that $\tau_n < \tau_N$. Let E^+ be the set of $n < N$ such $\tau_N < \tau_n$. Then $E^- < \tau_N < E^+$, both E^+ and E^- are finite, and at least one of them is non-empty. Let F^- be the set $\{g_{\tau_n} | n \in E^-\}$, and let F^+ be the set $\{g_{\tau_n} | n \in E^+\}$. Then $F^- < F^+$, by (i), and clearly F^- and F^+ are finite. Since G is a D-group, there are g in G with $F^- \lesssim g < F^+$.

If F^+ is not empty, we select g_{τ_N} as an arbitrary g such that $F^- < g < F^+$. Such a g is automatically less than any element of H.

If F^+ is empty, we choose g_{τ_N} as an arbitrary g such that $F^- \cup \{\gamma_o, \ldots, \gamma_N\} < g < H$. It is clear that such g exist, since $F^- \cup \{\gamma_o, \ldots, \gamma_N\}$ is a finite subset of $\{\gamma_n\}_{n<\omega}$.

With this choice of g_{τ_N}, we see easily that (i), (ii), (iii) now hold with N replaced by $N + 1$. Thus, by induction, we define $\{g_{\tau_n}\}_{n<\omega}$ so that (i), (ii), (iii) hold for all N. Thus we have defined $\{g_\mu\}_{\mu<\lambda}$, and it is clear that conditions (a) and (b) hold. We now show that (c) holds.

Suppose $\gamma \in G$ and $\gamma <_G h$ for all h in H. Now, for some n, $\gamma = \gamma_n$. Suppose that for all $k > n$ there is an $m < k$ such that $g_{\tau_k} \leq_G g_{\tau_m}$. Then by induction one sees that, for all integers ℓ, $g_{\tau_\ell} \leq_G \max_{0 \leq j \leq n} g_{\tau_j}$. But then $\{g_{\tau_\ell}\}_{\ell<\omega}$ has a maximum element, contrary to (a) and the fact that λ is a limit ordinal. We have thus shown that there is a $k_0 > n$ such that for all $m < k_0$ we have $g_{\tau_m} <_G g_{\tau_{k_0}}$.
But then by condition (iii), $g_{\tau_{k_0}} \geq \max_{0 \leq j \leq k_0} \gamma_j \geq \gamma_n = \gamma$. Thus $g_{\tau_{k_0}} \geq \gamma$, and since λ is a limit ordinal there is an integer m such that $g_{\tau_m} > \gamma$. Thus condition (c) is satisfied.

This concludes Part 2.

Part 3. Now let λ be a countable ordinal closed under $+$, $\cdot$, P and q. From the definition of P and q it is clear that there are such λ. Let L be $\mathbb{R}((t^G))_\lambda$. Then, by 10.25, L is real-closed. Let K be $\mathbb{R}((t^G))$. Then, by 10.26, K is real closed.

We will show that (K,L) is a model of $\mathcal{M}_s^+$, and that for (K,L) the associated system $\langle \mathcal{H}^0, \subset \rangle$ is isomorphic to $\langle X, <_X \rangle$.

Since $\mathbb{R}$ is archimedean, we may identify v^Z (on K) with the canonical valuation of K onto G. Since $v^Z[K] = v^Z[L] = G$, it is clear that (K,L) is a model of $\mathcal{A}_s$. By 22.7, we see that if $x \in K$, then $\Gamma(x) \in \mathcal{H}^0$ if and only if there is a proper convex subgroup H of G such that $\Gamma(x) = \{y \mid (\exists \gamma \in H)(v^Z(y) \geq \gamma)\}$. We know by Part 1 that G has exactly n proper convex subgroups. Thus the cardinality of $\mathcal{H}^{\circ}$ is at most n. We show that $\mathcal{H}^{\circ}$ has cardinality n.

Let H be a fixed proper convex subgroup of G. Then according to Part 2 there is a monotone increasing series $\{g_\mu\}_{\mu<\lambda}$ such that

$$\{g_\mu\}_{\mu<\lambda} <_G H ,$$

and such that for no g in G do we have

$$\{g_\mu\}_{\mu<\lambda} <_G g <_G H .$$

Let x be the element $\sum_{\mu<\lambda} t^{g_\mu} + t^{\circ}$ of K. We claim

$$\Gamma(x) = \{y \mid (\exists\, h \in H)(v^Z(y) \geq_G h)\} .$$

Suppose $\ell \in L$ and $h \in H$. Then we cannot have $v^Z(x-\ell) \geq_G h$. For, since $\{g_\mu\}_{\mu<\lambda} <_G H$, this would entail that each g_μ was in the support of ℓ, so that $\|\ell\| \geq \lambda$, contrary to the definition of L. Thus if $\ell \in L$ and $h \in H$ then $v^Z(x-\ell) <_G h$.

Suppose on the other hand that $\gamma <_G g_{\mu_o}$. Let ℓ be $\sum_{\mu\leq\mu_o} t^{g_\mu}$. Then $\ell \in L$ and $v^Z(x-\ell) >_G g_{\mu_o} >_G \gamma$. Thus if $\gamma <_G H$, then there is an ℓ in L such that $v^Z(x-\ell) >_G \gamma$.

It follows now from Lemma 22.5 that

$$\Gamma(x) = \{y \mid (\exists\, h \in H)(v^Z(y) \geq_G h)\} .$$

Since H was an arbitrary proper convex subgroup of G, it follows by 22.7 that to distinct proper convex subgroups of G there correspond distinct elements of $\mathcal{H}^{\circ}$. Thus $\mathcal{H}^{\circ}$ has cardinality n.

It follows that $\langle \mathcal{H}^{\circ}, \subset \rangle$ is isomorphic to $\langle X, <_X \rangle$ and the lemma is proved.

<u>Theorem 22.13</u>. $\mathcal{A}_s^+$ has at least $\aleph_0$ complete consistent extensions.

<u>Proof</u>. It is clear that for each positive integer n there is a sentence h_n of $\mathcal{L}$ which holds in a model of $\mathcal{A}_s^+$ if and only $\mathcal{H}^{\circ}$ has cardinality n. By 22.12, $\mathcal{A}_s^+ \cup \{h_n\}$ is consistent.

Section 23. Conclusion: Some Open Problems

1. Are there any other axiomatizable types of pairs (K,L) other than those corresponding to Classes A, B, C and D? If there are, a reasonable place to look for them would be in the case $S = \{0\}$, but the final section of the dissertation shows that this case is quite complicated.

2. Is the elementary theory of pairs of real-closed fields decidable? Is the elementary theory of pairs of D-groups decidable? In either case, a decision procedure would give a new decision procedure for the theory of an arbitrary linear order.

3. If L_1 is cofinal in K_1, are there at most finitely many elementary types of pairs (K,L) such that $\mathrm{clos}[(K,L)] \equiv (K_1,L_1)$?

4. For fixed L, are there only finitely many elementary types of pairs (K,L)? (For certain K, e.g., a non-principal ultraproduct of all countable real-closed fields, there are $2^{\aleph_0}$ elementary types of pairs (K,L).)

5. If $(K_1,L) \subseteq (K_2,L)$, and $(K_1,L) \equiv (K_2,L)$, does it follow that $(K_1,L) \prec (K_2,L)$?

6. If we assume the generalized continuum hypothesis, then we can prove, for regular α, the existence of η_α real-closed fields of cardinality $\aleph_\alpha$. Is it true that all pairs (K,L), where K is η_α of cardinality $\aleph_\alpha$, and L is η_β of cardinality $\aleph_\beta$, and $0 < \beta < \alpha$, are

elementarily equivalent? It is easy to see that for fixed α, β all such pairs are isomorphic. If there is in fact only one such type, can we characterize it in an elementary way? For such a type, L is not cofinal in K, nor is L dense in $\mathcal{R}_L$.

7. It would be interesting to investigate the theory of pairs of p-adic fields. It appears that some of the ideas and methods of this work could be used. Note that it is trivial to see that one can interpret within the theory of pairs of p-adic fields the theory of pairs of Z-groups.

BIBLIOGRAPHY

1. N. Alling, "A characterization of Abelian η_α-groups in terms of their natural valuation," Proc. Nat. Acad. Sci. U.S.A., 47 (1961), 711-713.
2. _______, "On the existence of real-closed fields that are η_α-sets of power $\aleph_\alpha$," Trans. Amer. Math. Soc., 103 (1962), 341-352.
3. E. Artin, "Algebraische Konstruktion reeller Körper," Abhandlungen des mathematischen Seminars der hamburgischen Universität, 5 (1926), 85-99.
4. _______, "Über die Zerlegung definiter Funktionen in Quadrate," Abhandlungen des mathematischen Seminars der hamburgischen Universität, 5 (1926), 100-115.
5. J. Ax, "The undecidability of power series fields," Proc. Amer. Math. Soc., 16 (1965), 846.
6. _______, Lecture notes on number theory, Cornell University, unpublished.
7. J. Ax and S. Kochen, "Diophantine problems over local fields: I," Amer. J. Math., 87 (1965), 605-630.
8. _______, "Diophantine problems over local fields: II," Amer. J. Math., 87 (1965), 631-648.
9. _______, "Diophantine problems over local fields: III," Annals of Mathematics, ser. 2, 83 (1966), 437-456.
10. R. Baer, "Über nicht - Archimedisch geordnete Körper," S. - B. Heidelberger Akad., 8 (1927), 3-13.
11. P. J. Cohen, "Decision procedures for real and p-adic fields," Stanford Univ., 1967, mimeographed.
12. A. Ehrenfeucht, "Decidability of the linear ordering relation," A.M.S. Notices, 6 (1959).
13. P. Erdös, L. Gillman and M. Henriksen, "An isomorphism theorem for real closed fields," Annals of Mathematics, ser. 2, 61 (1955), 542-554.
14. Ju. L. Ersov, "Undecidability of certain fields," (Russian), Doklady Akad. Nauk SSSR, 161 (1965). Translated in Soviet Mathematics (1965), 349-352.

5. Ju. L. Ersov, "On the elementary theory of maximal normed fields," (Russian), Doklady Akad. Nauk SSSR, 165 (1965). Translated in Soviet Mathematics (1965), 1390-1393.

6. S. Feferman and R.L. Vaught, "The first order properties of products of algebraic systems," Fundamenta Mathematicae, 47 (1959), 57-103.

7. T. Frayne, A. Morel and D. Scott, "Reduced direct products," Fundamenta Mathematicae, 51 (1962), 195-228.

L. Fuchs, Partially ordered algebraic systems, Pergamon Press, Oxford-London-New York-Paris, 1963.

L. Gillman and M. Jerison, Rings of continuous functions, Princeton-Toronto-London, New York, 1960.

K. Gödel, "The consistency of the continuum hypothesis," Annals of Mathematics Studies, Princeton University Press, 1940.

H. Hahn, "Über die nichtarchimedischen Grössensysteme," S.-B. Akad. Wiss. Vienna, 116 (1907), 601-653.

F. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914.

N. Jacobson, Lectures in Abstract Algebra, Vol. III, Van Nostrand, 1964.

I. Kaplansky, "Maximal fields with valuation," Duke Mathematical Journal, 9 (1942), 303-321.

_______, "Maximal fields with valuation: II," Duke Mathematical Journal, 12 (1945), 243-248.

H. J. Keisler, "Ultraproducts and elementary classes," Indag. Math., 64 (1961), 477-495.

_______, "Complete theories of algebraically closed fields with distinguished subfields," Michigan Mathematical Journal, 11 (1964), 71-81.

S. Kochen, "Ultraproducts in the theory of models," Annals of Mathematics, ser. 2, 74 (1961), 221-261.

W. Krull, "Allgemeine Bewertungstheorie," J. Reine u. Angew. Math., 167 (1932), 160-196.

S. Lang, "The theory of real places," Annals of Mathematics, ser. 2, 57 (1953), 378-391.

. H. Lauchli and J. Leonard, "On the elementary theory of linear order," Fundamenta Mathematicae, 59 (1966), 109-116.

. M. Morley and R.L. Vaught, "Homogeneous universal models," Math. Scand., 11 (1962), 37-57.

. A. Ostrowski, "Zur arithmetischen Theorie der algebraische Grössen," Göttinger Nachrichten (1919), 279-298.

. A. Robinson, Complete Theories, Studies in Logic, North Holland, 1956.

. _______, "Solution of a problem of Tarski," Fundamenta Mathematicae, 47 (1959), 179-204.

. _______, "Model theory and non-standard arithmetic," Infinitistic Methods, Warsaw, 1959, 265-302.

. _______, Introduction to Model Theory and to the Metamathematics of Algebra, Studies in Logic, North Holland, 1963.

. _______, Non-Standard Analysis, Studies in Logic, North Holland, 1966.

. A. Robinson and E. Zakon, "Elementary properties of ordered abelian groups," Trans. Amer. Math. Soc., 96 (1960), 222-236.

. J. Robinson, "The Decision Problem for Fields," Model Theory, Studies in Logic, North Holland.

. O.F.G. Schilling, The Theory of Valuations, Mathematical Surveys No. IV, American Mathematical Society.

. D. Scott, "On completing ordered fields," to appear.

. R. Sikorski, "On an ordered algebraic field, Soc. Sci. Lett. Varsovie C.R. Cl. III Sci. Math. Phys., 41 (1948), 69-96.

. R. Sikorski, "On algebraic extensions of ordered fields," Ann. Soc. Polon. Math., 22 (1949), 173-184.

. E. Steinitz, Algebraische Theorie der Körper, Berlin, 1930.

. W. Szmielew, "Elementary properties of Abelian groups," Fundamenta Mathematicae, 41 (1954), 203-271.

. A. Tarski, "Some notions and methods on the borderline of algebra and metamathematics," Proc. of the Int. Cong. of Math., Vol. I, Cambridge 1950, 705-720.

A. Tarski, Logic, Semantics and Metamathematics, Clarendon Press, 1956.

A. Tarski and J.C.C. McKinsey, A Decision Method for Elementary Algebra and Geometry, Rand Corporation, Santa Monica, 1948.

A. Tarski, A. Mostowski and R.M. Robinson, Undecidable Theories, Studies in Logic, North Holland, 1953.

A. Tarski and R.L. Vaught, "Arithmetical extensions of relational systems," Compositio Math., 13 (1957), 81-102.

B. Van der Waerden, Einführung in die Algebraische Geometrie, Springer, 1939.

______, Modern Algebra, Vol. 1, Revised English edition, Ungar, New York, 1949-50.

Notes. The following are some bibliographical references for specific sections of the dissertation.

Introduction. Steinitz's results are in [45]. Tarski's are in [49], and Cohen has a simplification in [11]. Robinson's results are in [35]. In [27], Keisler extended Robinson's results about algebraically closed fields.

Section 1. See [3], [4], [6], [18], [23] and [37].

Section 3. Lemma 3.4 is from [42]. Lemma 3.8 is from [13].

Section 4. See [18].

Section 5. See [1], [2], [18], [23], [29], [30] and [41].

Section 6. See [1], [2], and [30].

Section 9. See [1], [2], and [18].

Section 10. See [1], [2], [6], [8], [18], [52], and [53].

Section 11. See Scott [42].

Section 12. See [23].

Section 13. For $\alpha > 0$, η_α-sets were introduced by Hausdorff [22], who showed that the existence of η_α sets of cardinality $\aleph_\alpha$ implies that α is regular and $\aleph_\alpha = \sum_{\lambda<\alpha} 2^{\aleph_\lambda}$. Conversely, if $\aleph_\alpha = \sum_{\lambda<\alpha} 2^{\aleph_\lambda}$, there exists an η_α set of cardinality $\aleph_\alpha$. The existence of η_1 D-groups and real-closed fields of cardinality $\aleph_1$, on the assumption of the continuum hypothesis, was first established by the theory of real-valued continuous functions on completely regular topological spaces. See [19]. The existence question for $\alpha > 1$, and assuming $\aleph_\alpha = \sum_{\lambda<\alpha} 2^{\aleph_\lambda}$, was settled affirmatively by two methods. Alling [2] used the theory of formal power-series. Keisler [26] used ultrapowers to prove a vastly more general result. The method of Morley and Vaught [32] establishes the same general result.

Hausdorff [22] proved 13.3. Erdös, Gillman and Henriksen proved 13.5 in [13], and Alling and Kochen (independently) proved 13.4 by a similar method. See Kochen [28] for metamathematical applications of these results. Kochen's method generalizes, by [26] and [32], and the papers [7], [8], [9] of Ax and Kochen, and [15] of Ersov, are beautiful examples of the method's power.

Section 14. See [6], [24] and [41] for more details about pseudo-Cauchy sequences.

Section 15. The theorem of Ax and Kochen needed here is in [6]. See also [7], [8], [9] and [15].

Section 18. For the requisite information about ultrafilters, see [6], [17], [26] and [28]. For details about the constructible universe, see Gödel [20].

Section 21. In Hausdorff [22] there is an example which shows that there are $2^{\aleph_0}$ elementary types of linear ordering. Ehrenfeucht [12] showed that the elementary theory of an arbitrary linear order is decidable. For a proof, see Läuchli and Leonard [31].