Model Theory and Differential Algebra

Coven–Wood Lecture II

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Differential Algebra

We work with fields (or rings) of characteristic zero $(K, +, \cdot)$ together with a derivation $\delta : K \to K$ such that

$$\delta(x + y) = \delta(x) + \delta(y)$$

$$\delta(xy) = x\delta(y) + y\delta(x).$$

We sometimes use $D(x)$ or x' rather than $\delta(x)$.

Examples i) trivial derivation $\delta(x) = 0$;
ii) $\mathbb{C}(x)$ where $\delta = \frac{d}{dx}$;
iii) Meromorphic functions on on an open set U with $\frac{d}{dx}$
iv) formal Laurent series $K((t))$ with $\delta(\sum a_i t^i) = \sum i a_i t^{i-1}$
or $\delta(\sum a_i t^i) = \sum \delta(a_i) t^i$.

Differential Polynomials

K a differential field

We build a ring of differential polynomials

$$K\{X_1,\ldots,X_n\} = K[X_1,\ldots,X_n,X_1',\ldots,X_n',X_1'',\ldots,X_n'',\ldots].$$

There is a unique way to extend δ to a derivation on $K\{\overline{X}\}$ such that $\delta(X_i^{(j)}) = X_i^{j+1}$.

An ideal $I \subset K\{\overline{X}\}$ is a differential ideal if $f \in I \Rightarrow \delta(f) \in I$.

A differential algebraic variety $V \subset K^n$ if there are $f_1,\ldots,f_m \in K\{\overline{X}\}$ such that $V = \{\overline{x} \in K^n : f_1(\overline{x}) = \cdots = f(\overline{x}) = 0\}$.

Ritt's program: Develop an algebraic theory of solutions to differential equations paralleling algebraic geometry.

The Basis Theorem

$K\{X\}$ is non-Noetherian .
Let I_n be the differential ideal generated by $X^2, (X')^2, \dots, (X^{(n)})^2$.
Then $I_0 \subset I_1 \subset I_2 \subset \dots$.

Theorem (Basis Theorem)

In $K\{X_1, \dots, X_n\}$ there are no properly increasing sequence of radical differential ideals.
In particular, every radical differential ideal is finitely generated.

Corollary

There are no proper infinite decreasing sequences $V_0 \supset V_1 \supset \dots$ of differential algebraic varieties.

The closed sets determine a Noetherian topology on K^n. (Kolchin topology)

Algebraic Geometry v. Differential Algebraic Geometry

Much of Ritt's early work was motivated by developing this analogy, but there are many basic ways in which AG and DAG are very different.

Good examples:
• Any radical differential ideal can be decomposed as a finite intersection of prime differential ideals.
• Forms of the Nullstellensatz.

Bad example: Let $V_n = \{X : X^{(n)} = 0\}$. Then $V_1 \subset V_2 \subset V_3 \subset \cdots \subset K$ is an infinite increasing sequence of irreducible differential algebraic varieties in K.

Successes of Differential Algebra

- Ritt's theory of general vs. special components
- Kolchin's development of differential Galois theory
- Kolchin school's development of differential algebraic group theory

Enter model theory

The theory of algebraically closed fields has a highly developed model theory.
Is there an analog for differential algebra?

What's a *differentially closed field?*

One important property of algebraically closed fields is that they are *existentially closed*, i.e., if K is algebraically closed, $f_1, \ldots, f_m \in K[X_1, \ldots, K_n]$, $K \subset L$ and there is $\overline{x} \in L^m$ such that $f_1(\overline{x}) = \cdots = f_m(\overline{x}) = 0$, there the is already a solution in K^n.

Can we axiomatize the theory of existentially closed differential fields?

Theorem (A. Robinson)

The theory of existentially closed fields is axiomatizable.

Makes heavy use of Ritt's theory of differential ideals. Needed to consider differential polynomials in many variables.

Blum's Thesis

The *order* of a differential polynomial in $K\{X\}$ is the largest n such $X^{(n)}$ occurs.

Let DCF be the theory of algebraically closed fields K of characteristic 0 such that:
If $f, g \in K\{X\}$ and $\mathrm{ord}(f) > \mathrm{ord}(g)$, then there is $x \in K$ such that $f(x) = 0$ and $g(x) \neq 0$.

Theorem (Blum)

- *DCF characterizes the existentially closed differential fields.*
- *DCF has quantifier elimination*

Definable sets = Boolean combinations of differential algebraic varieties.

Embarrassing Question: Examples of DCF?

The Differential Nullstellensatz

Over a differentially closed field K there is a bijective Galois correspondence between radical differential ideals and differential algebraic varieties.

Let $I \subset K\{X_1, \ldots, X_n\}$
$I \mapsto V(I) = \{\overline{x} \in K^n : f(\overline{x}) = 0 \text{ for all } f \in I\}$.

Main Lemma

Lemma

Let K be a differentially closed field. Let $I \subset K\{\overline{X}\}$ be a radical differential ideal and suppose $g \in K\{\overline{X}\} \setminus I$. There is $\overline{x} \in V(I)$ such that $g(\overline{x}) \neq 0$.

Proof.

There is a prime differential ideal $P \subset K\{X\}$ such that $I \subseteq P$ and $g \notin P$. There are $f_1, \ldots, f_m$ generating I.

Let $L \supset K$ be the quotient field of $K\{\overline{X}\}/P$. Let $x_i = X_i/P \in L^n$. Then $f_1(\overline{x}) = \cdots = f_m(\overline{x}) = 0$ and $g(\overline{x}) \neq 0$.
By existential closedness, there is $\overline{x} \in K^n$ with $f_1(\overline{x}) = \cdots = f_m(\overline{x}) = 0$ and $g(\overline{x}) \neq 0$. Note $\overline{x} \in V(I)$. □

Types and ω-stability

Let $K \subset L$ be differentially closed fields and let $\overline{a} \in L^n$. The type of $\overline{a}$ over K is

$$\mathrm{tp}(\overline{a}/K) = \{\phi(\overline{a}) : L \models \phi(\overline{a}), \phi(\overline{v}) \text{ a formula with parameters from } K\}.$$

By quantifier elimination, $\mathrm{tp}(\overline{a}/K)$ is determined by the prime differential ideal

$$I_{\overline{a}} = \{f \in K\{X_1, \ldots, X_n\} : f(\overline{a}) = 0\}.$$

By the Basis Theorem $I_{\overline{a}}$ is finitely generated.

Corollary (Blum)

(ω-stability) There are only $|K|$ types over K.

This is a big deal

There is a well developed and powerful theory for ω-stable theories.

Payoffs from ω-stability

Definition

Let k be a differential field. We say that a DCF $K \supseteq k$ is a *differential closure* of k if for any DCF $L \supset k$, there is a differential embedding $\sigma : K \to L$ fixing k.

Theorem

- *(Blum) Every differential field k has a differential closure.*
- *(Shelah) If K_1 and K_2 are differential closures of k, then they are isomorphic over k.*

There is a deep well developed theory of ω-stable groups. Pillay has applied this to answer many longstanding open questions in Kolchin's theory of differential algebraic groups and to differential Galois theory.

Carol Wood's contributions

 Carol Wood made a number of early contributions to the model theory of differential fields. In particular, develop the analog of the theory of differentially closed fields of characteristic $p > 0$ and proved existence of differential closures (without ω-stability).
Later, Carol worked with Margit Messmer on separably closed fields with a sequene of Hasse derivations.

Her student Tracy McGrail did important work on differentially closed fields with several commuting derivations.

Strongly minimal sets

We briefly leave DCF to introduce a fundamental notion in ω-stability. Let $\mathcal{M}$ be a (sufficiently rich) $\mathcal{L}$-structure.

Definition

We say that a definable $X \subset \mathcal{M}^n$ is *strongly minimal* if X is infinite and for any definable $Y \subset X$ either Y or $X \setminus Y$ is finite.

Examples

- Equality: $\mathcal{M}$ an infinite set with no structure and $X = \mathcal{M}$.
- Successor: $\mathcal{M}$ an infinite set $f : \mathcal{M} \to \mathcal{M}$ a bijection with no finite orbits, $X = \mathcal{M}$.
- DAG: $\mathcal{M}$ a torsion free divisible abelian group, $X \subseteq \mathcal{M}^n$ a translate of a one-dimensional subspace defined over $\mathbb{Q}$.
- ACF: K an algebraically closed field and $X \subseteq K^n$ an irreducible algebraic curve ($\pm$ finitely many points).

Model Theoretic Algebraic Closure

Definition

If $a \in \mathcal{M}$, $B \subset \mathcal{M}$, a is *algebraic* over B if there is an $\mathcal{L}$-formula $\phi(x, y_1, \ldots, y_m)$ and $\mathrm{b} \in B^m$ such that $\phi(a, \mathrm{b})$ and $\{x \in \mathcal{M} : \phi(x, \mathrm{b})\}$ is finite.
Let $\mathrm{cl}(B) = \{a : a \text{ algebraic over } B\}$.

- equality: $\mathrm{cl}(A) = A$.
- Successor: $\mathrm{cl}(A) = \bigcup_{a \in A}$ orbit of a
- DAG: $\mathrm{cl}(A) = \mathrm{span}_{\mathbb{Q}}(\mathrm{A})$.
- ACF: $\mathrm{cl}(A)=$ algebraic closure of field generated by A.

Shelah Philosophy To understand finite dimensional sets in an ω-stable theories we must underdstand strongly minimal sets and the algebraic relations between them.

Combinatorial Geometry of Strongly Minimal Sets

Definition

A strongly minimal set X is *trivial* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$ for all $A \subseteq X$.

Equality and Successor are trivial

Definition

A strongly minimal set X is *modular* if $c \in \mathrm{cl}(B \cup \{a\})$, then $c \in \mathrm{cl}(b, a)$ for some $b \in B$, for all $a \in X$, $B \subseteq X$.

DAG is non-trivial modular: If $c = \sum m_i b_i + na$ where $m_i, n \in \mathbb{Q}$, then $c = b + na$ where $b = \sum m_i b_i$.

ACF is non-modular
Consider $a_0, \ldots, a_{n-1}$ algebraically independent and $x^n + \sum a_i x^i = 0$.

When are two strongly minimal sets "the same"?

Definition

Two strongly minimal sets X and Y are *non-orthogonal* ($X \not\perp Y$) if there is a definable $R \subseteq X \times Y$ such that $\{y \in Y : (x, y) \in R\}$ is non-empty finite for all but finitely many $x \in X$.

Idea: non-orthogonal = "intimately related", orthogonal = "not related".

Goal Try to understand strongly minimal sets up to orthogonality.

In ACF: If X is a curve there is $\rho : X \to K$ rational so $X \not\perp K$.
So in ACF K is the only strongly minimal set up to $\not\perp$.

Strongly minimal sets in DCF

In DCF $\mathrm{cl}(A)$ is the field theoretic algebraic closure differential field generated by $\mathbb{Q}(A)$.

Early examples:

• The field of constants $C = \{x : \delta(x) = 0\}$ is non-modular strongly minimal–indeed any definable $X \subset C^n$ is definable in the field structure on C.

• Any irreducible order 1 equation $f(X) = 0$ is strongly minimal. Rosenlicht, Kolchin and Shelah gave examples that are geometrically trivial. For example, $X' = X^3 - X^2$.

• Poizat gave an example $X''/X' = \frac{1}{X}$ of an order 2 strongly minimal equation. This must be $\perp C$.

Manin kernels

Let A be a simple abelian variety that is not isomorphic to one defined over the constants C.

Let $A^\sharp$ be a minimal infinite definable subgroup of A. $A^\sharp$ is the closure in the Kolchin topology of $\mathrm{Tor}(A)$.

Theorem (Hrushovski–Sokolović)

In differentially closed fields:

- $A^\sharp$ *is a locally modular strongly minimal set.*
- *Moreover, if* X *is a non-trivial locally modular strongly minimal set, then* $X \not\perp A^\sharp$*, for some Manin kernel* $A^\sharp$*.*
- $A^\sharp \not\perp B^\sharp$ *if and only if there is an isogeny* $f : A \to B$*.*

Digression–A Diophantine Application

Let A be a simple abelian variety of dimension $d \geq 2$ and let $C \subset A$ be a curve. Consider $C \cap \mathrm{Tor}(A)$.

Manin–Mumford Conjecture: $C \cap \mathrm{Tor}(A)$ is finite.

Suppose A is not defined over the constants.
Then $A^\sharp$ is strongly minimal and $A^\sharp$ contains Tor(A).
If $A^\sharp \cap C$ is infinite, then $A^\sharp \setminus C$ is finite and all but finitely many torsion points are in C and C is Zariski dense in A, a contradiction.

This was a key insight in Hrushovski's proof of the Mordell–Lang conjecture for function fields.

Zilber's Principle

What about non-locally modular strongly minimal sets?

Theorem (Hrushovski–Sokolović)

If X is strongly, minimal and not locally modular, then $X \not\perp C$.

So, essentially, C is the only non-locally modular strongly minimal set.

This was originally proved using the Zariski geometry machinery of Hrushovski and Zilber, but later given a proof by Pillay and Ziegler avoiding Zariski geometries.

What about trivial strongly minimal sets?

Trivial examples

- Poizat's example $\frac{x''}{x'} = \frac{1}{x}$.

- Transcendence degree one examples

  - McGrail, following Rosenlicht, characterized when $X' = f(X)$ is trivial for $f(X) \in C(X)$.
  - Hrushovski and Itai found strongly minimal examples on curves of genus> 1 defined over C.

- Nagloo and Pillay showed that generic Painlevé equations are strongly minimal and trivial, for example

$$P_{II}(\alpha): \;\; X'' = 2X^3 + tX + \alpha$$

is trivial strongly minimal, where $\alpha \in C$ is transcendental and $t' = 1$. They also studied algebraic relations between solutions to different Painlevé equations.

• Freitag and Scanlon proved that the third order differential equation satisfied by the *j*-function is trivial strongly minimal. This was the first non $\aleph_0$-categorical example. This built on work of Pila in transcendence theory.

This has been greatly extended by recent work of Blázquez-Sanz, Casale, Freitag and Nagloo on the differential equations satisfied by uniformizing functions for Fuchsian groups.
Their work can be used to prove new transcendence results in the style of Pila and his collaborators.

It can be hard to find strongly minimal sets and to prove they are trivial.

Ubiquity of strongly minimal sets

Theorem (Jaoui)

Generic planar vector fields

$$x' = p(x, y)$$
$$y' = q(x, y)$$

$\max(deg(p), deg(q)) \geq 3$, *over the constants give rise to transcendence degree 2 trivial strongly minimal sets*

Theorem (DeVilbis–Freitag)

If $f(X)$ is a generic differential polynomial of order $n > 1$ and degree $d \geq 6$, then $f(X) = 0$ is strongly minimal.

Poizat's Example

Let $V = \{X : X' = XX''\}$

Theorem (Poizat)

The only infinite irreducible differential algebraic subvariety of V is given by $X' = 0$.

Corollary

$\frac{X''}{X'} = \frac{1}{X}$ *defines a trivial strongly minimal set.*

Folklore: Any strongly minimal set defined over C of order at least 2 is trivial.

order 2 $\Rightarrow \perp C$;

defined over $C \Rightarrow \perp A^\sharp$ for any Manin kernel;

Generalized Poizat Equations

Joint work with Jim Freitag, Rémi Jaoui and Ronnie Nagloo.

For $f(z) \in C(z)$ consider the differential equation $\frac{z''}{z'} = f(z)$.
Let V_f denote the solutions.

Theorem

V_f is strongly minimal if and only there is no $g \in C(z)$ with $f = \frac{dg}{dz}$.

$\Rightarrow$ If $f = \frac{dg}{dz}$ and $z' = g(z) + c$ for some $c \in C$, then $z'' = f(z)z'$.
Thus there is an infinite family of order 1 differential algebraic subvarieties of V_f.

Suppose $f(z) \in C(z)$ has no antiderivative in $C(z)$.

Partial Fractions Any $f(z) \in C(z)$ can be expressed

$$f(z) = g'(z) + \sum_{i=1}^{n} \frac{c_i}{z - \alpha_i}$$

where $g(z) \in C(z)$, and $c_1, \ldots, c_n, \alpha_1, \ldots, \alpha_n \in C$.

f has an antiderivative in $C(z)$ if and only if $n = 0$.

Suppose $f(z)$ has no an antiderivative.
By a change of variables, we may assume some $\alpha_i = 0$.
Consider the power series expansion

$$f(z) = \sum_{n=m}^{\infty} a_n z^m$$

then $a_{-1} \neq 0$. We call a_{-1} the *residue* at 0.

If V_f is not strongly minimal, then we can find a differential field (K, δ) and $z \in V_f$ transcendental over K such that z and z' are algebraically dependent over K.

Consider the Puiseux series field $K\langle\langle z \rangle\rangle = \bigcup_n K((z^{\frac{1}{n}}))$.

Since this field is algebraically closed we can identify z' as some series u in $K\langle\langle z \rangle\rangle$

Define a derivation D on $K\langle\langle z \rangle\rangle$ such that

$$D\left(\sum a_i z^i\right) = \sum \delta(a_i) z^i + u \sum i a_i z^{i-1}$$

D extends the natural derivation on $K(z, z')$.

Let $z' = u = \sum_{i=0}^{\infty} a_i z^{r+\frac{i}{n}}$, where $v(u) = r$.
Then

$$z'' = D(u) = \sum \delta(a_i) z^{r+\frac{i}{n}} + u \sum \left(r + \frac{i}{n}\right) a_i z^{r+\frac{i}{n}-1}$$

Since

$$v\left(\sum \delta(a_i) z^{r+\frac{i}{n}}\right) \geq r,$$

$$f(z) = \frac{z''}{z'} = \frac{D(u)}{u} = \alpha + \sum \left(r + \frac{i}{n}\right) a_i z^{r+\frac{i}{n}-1}$$

where $v(\alpha) \geq 0$.

The coefficient of z^{-1} on the right summand is 0.
Thus we can not have $\frac{z''}{z'} = f(z)$, a contradiction.

Algebraic relations between solutions

Suppose $f, g \in C(z)$ have no antiderivatives in C (possibly $f = g$). Suppose $K \supseteq C$ is a differential field $x \in V_f$, $y \in V_g$ are each transcendental over K.

Theorem

If $K(x, x')^{\mathrm{alg}} = K(y, y')^{\mathrm{alg}}$, then $C(x)^{\mathrm{alg}} = C(y)^{\mathrm{alg}}$.

In particular, if $y \in \mathrm{cl}(x)$, then $y \in C(x)^{\mathrm{alg}}$.

Theorem

i) If $f \neq g$ and $V_f \not\perp V_g$, then $g = f \circ \phi$ for some affine transformation $\phi(x) = ax + b$;
ii) If $x, y \in V_f$ and $y \in \mathrm{cl}(x)$, there is ϕ as above with $f = f \circ \phi$ and $\phi(x) = y$.

Thank You!

Ethan Coven & Carol Wood