# The Model Theory of Differential Closures

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# Differential Fields

A *differential field* is a field $k$ of characteristic 0 with a derivation $D : k \to k$

$$D(a+b) = D(a) + D(b) \text{ and } D(ab) = aD(b) + bD(a).$$

The constant subfield of $k$ is $C_k = \{x \in k : D(x) = 0\}$.

Examples:

- $k(t)$ with $d/dt$, $C_{k(t)} = k$
- $\mathcal{M}$ germs of complex meromorphic functions at 0, $C_{\mathcal{M}} = \mathbb{C}$.
- any field $k$ with the trivial derivation $D(x) = 0$, $C_k = k$.

# Differential Polynomials

A *differential polynomial* in variables $X_1, \ldots, X_n$ over $k$ is an element of the ring $k\{X_1, \ldots, X_n\}$ which is

$$k[X_1, \ldots, X_n, D(X_1), \ldots, D(X_n), \ldots, D^{(m)}(X_1), \ldots, D^{(m)}(X_n), \ldots].$$

The *order* of $f \in K\{X_1, \ldots, X_n\}$ is the largest $m$ such that some $D^{(m)}$ occurs.

We extend $D$ to a derivation on $k\{X_1, \ldots, X_n\}$ using the product rule. For example

$$D(aX^2Y') = D(a)X^2Y' + 2aXX'Y' + aX^2Y''$$

# Differential Ideals

An ideal $I \subset k\{X_1, \ldots, X_n\}$ is a *differential ideal* if $D(f) \in I$, whenever $f \in I$.

**Theorem (Ritt–Raudenbush Basis Theorem)**

*There are no infinite ascending sequence of radical differential ideals in $k\{X_1, \ldots, X_n\}$.*

**Corollary**

*Every radical differential ideal is finitely generated.*

# What is a differentially closed field?

We would like to find a differential analog *algebraically closed fields*.

*What is an algebraically closed field?*

**Axiomatization**

- axioms for fields;
- $\forall c_0 \ldots \forall c_{n-1} \exists x \; x^N + c_{N-1}x^{N-1} + \cdots + c_1 x + c_0 = 0$ for $n = 2, 3, \ldots$.

**Key Property of Algebraically Closed Fields:** *Existentially Closed*

If $K$ is an algebraically closed field, $f_1, \ldots, f_m, g \in K[X_1, \ldots, X_n]$ and $L \supset K$ is a field containing a solution to

$$f_1(\mathbf{X}) = \cdots = f_m(\mathbf{X}) = 0 \wedge g(\mathbf{X}) \neq 0,$$

then there is already a solution in $K$.

# What is a differentially closed field

*Can we axiomatize the class of existentially closed differential fields*?

First Answer: (A. Robinson) yes!

Robinson's axiomatization uses heavily the Ritt–Kolchin theory of differential ideals in polynomial rings in several variables.

*Can we give an axiomatization using only differential polynomials in one variable*?

Second Answer: (L. Blum) yes!

# $\mathrm{DCF}_0$

Consider the theory DCF of algebraically closed differential fields $K$ of characteristic zero such that if $f, g \in K\{X\}$ and $\mathrm{ord}(f) > \mathrm{ord}(g)$, then there is $x \in K$ with

$$f(x) = 0 \wedge g(x) \neq 0.$$

**Fact**: If $k$ is a differential field and $\mathrm{ord}(f) > \mathrm{ord}(g)$, there is a differential field $l \supset k$ and $a \in l$ such that $f(a) = 0$ and $g(a) \neq 0$. Thus every field extends to a model of DCF.

**Theorem (Blum)**

*The theory* $\mathrm{DCF}$ *has quantifier elimination–i.e. every formula* $\phi(\mathbf{x})$ *is equivalent to a finite boolean combination of formulas* $f(\mathbf{x}) = 0$.

**Corollary**

*The theory* $\mathrm{DCF}$ *is complete and decidable.*

# Existential Closure and DCF

Suppose $K, L \models DCF$, $f_1, \ldots, f_m \in K\{X_1, \ldots, X_n\}$, $K \subset L$ and

$$L \models \exists \mathbf{x}\ f_1(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = 0$$

We think of $\exists \mathbf{x}\ f_1(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = 0$ as a sentence $\psi(\mathbf{c})$ where $\mathbf{c}$ are the elements of $K$ appearing as coefficients in $f_1, \ldots, f_m$.
By quantifier elimination there is a quantifier free formula $\theta(\mathbf{w})$ such that

$$\psi(\mathbf{w}) \Leftrightarrow \theta(\mathbf{w}).$$

For quantifier free formulas $K \models \theta(\mathbf{c}) \Leftrightarrow L \models \theta(\mathbf{c})$. Thus $K \models \theta(\mathbf{c})$ and

$$f_1(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = 0$$

has a solution in $K$.

Thus DCF axiomatizes existentially closed differential fields.

# Natural Examples?

**Embarrassing Question** *What is a natural example of a differentially closed field?*

Unfortunately there are none.

**Theorem (Seidenberg's Embedding Theorem)**

*Every countable differential field is isomorphic to a field of germs of complex meromorphic functions.*

*Why study differentially closed fields?*

- They are a natural context for studying differential algebra and differential Galois theory;
- They are useful in number theoretic applications (Buium/Hrushovski);
- They are a rich source of interesting model theoretic phenomena.

# What's a differential closure?

*What's an algebraic closure?*

- (existence) For any field $k$ there is $k^{\mathrm{alg}} \supseteq k$ algebraically closed, such that if $K \supset k$ is an algebraically closed field, then there is $j : k^{\mathrm{alg}} \to K$ an embedding fixing $k$.
- (uniqueness) Any algebraic closure of $k$ is isomorphic to $k^{\mathrm{alg}}$ over $k$.
- (minimality) There is no algebraically closed field $K$ with $k \subseteq K \subset k^{\mathrm{alg}}$.

# Types

Let $k \subset K$ be differential fields $\mathbf{a} \in K^n$.

The *type* of $\mathbf{a}$ over $k$

$$\mathrm{tp}(\mathbf{a}/k) = \{\phi(x_1, \ldots, x_n) : K \models \phi(\mathbf{a}), \phi \text{ has parameters from } k\}.$$

$S_n(k) = \{p : p \text{ a } n\text{-type over } k \text{ realized in some } K \supseteq k\}$.

Let $I_{\mathbf{a}/k} = \{f \in k\{\mathbf{X}\} : f(\overline{a}) = 0\}$. Note that this is a prime differential ideal.

By quantifier elimination $I_{\mathbf{a}/k}$ completely determines $\mathrm{tp}(\mathbf{a}/k)$ and there is a bijection between $S_n(k)$ and $\mathrm{Spec}_D(k\{\mathbf{X}\})$, the space of differential prime ideals.

$\omega$-*stability*: (Blum) By the Ritt–Raudenbush Basis Theorem every prime ideal in $k\{\mathbf{X}\}$ is finitely generated. Thus $|S_n(k)| = |k|$.

# The Stone Topology

*Stone Topology*: We topologize $S_n(k)$ taking subbasic open sets of the form $[f = 0] = \{p : \text{“}f = 0\text{”} \in p\}$ and $[g \neq 0] = \{p : \text{“}g \neq 0\text{”} \notin p\}$.
The Stone topology is compact.

**Key Model Theoretic Insight** The importance of isolated points in $S_n(k)$.

$p \in S_n(k)$ is isolated and $K \supseteq k$ is differentially closed. Then $p$ is realized in $k$.

**Lemma**

*The isolated points in $S_n(k)$ are dense.*

Suppose $U$ is a open set containing no isolated points. We can find $p, q \in U$ and $f \in k\{X\}$ such that “$f(x) = 0$” $\in p$ and “$f(x) \neq 0$” $\in q$.
Let $U_1 = U \cap [f = 0]$, $U_2 = U \cap [f \neq 0]$. These are open, nonempty and contain no isolated points.
Iterating, build a perfect binary tree and find a countable differential field $l$, where $|S_n(l)| = 2^{\aleph_0}$.

# A constructible closure

Let $k \subset K$ where $K$ is differentially closed.
A *construction sequence* over $k$ is a sequence $(a_\alpha : \alpha < \delta)$, $a_\alpha \in K$, such that for all $\alpha$:

- $a_\alpha \notin k\langle a_\beta : \beta < \alpha\rangle$.
- $\mathrm{tp}(a_\alpha / k\langle a_\beta : \beta < \alpha\rangle)$ is isolated in $S_1(k\langle a_\beta : \beta < \alpha\rangle)$.
- No element of $K \setminus k\langle a_\alpha : \alpha < \delta\rangle$ realizes an isolated type over $k\langle a_\alpha : \alpha < \delta\rangle$

Let $k^* = k\langle a_\alpha : \alpha < \delta\rangle$.

**claim** $k^*$ is differentially closed.

Suppose $f, g \in k^*\{X\}$ with $\mathrm{ord}(f) > \mathrm{ord}(g)$.
Since isolated types are dense there is an isolated type $p \in S_1(k^*)$ and $a \in K$ realizing $p$ with $f(a) = 0 \wedge g(a) \neq 0$.
But then $a \in k^*$. Thus $k^* \models$DCF.

# Existence of Differential Closures

## Definition

We say that $K$ is a *differential closure of* $k$ if $k \subseteq K$ is differentially closed and for any differentially closed $L \supseteq k$, there is a differential embedding $j : K \to L$ fixing $k$.

## Lemma

*$k^*$ is a differential closure of $k$.*

Inductively we build embeddings $f_\beta : k\langle a_\alpha : \alpha < \beta\rangle \to L$.

- $f_0 : k \to k$ is the identity;
- $f_\beta = \bigcup_{\alpha<\beta} f_\alpha$ for $\beta$ a limit ordinal;
- $a_\beta$ realizes an isolated type over $k\langle a_\alpha : \alpha < \beta\rangle$, there must be $b \in L$ realizing the image of that type under $f_\beta$, define $f_{\beta+1}$ by $a_\alpha \mapsto b$

This argument is due to Morley for $\omega$-stable theories. Blum realized that it applied to differential fields.

# Some properties of differential closures

We denote $k^*$ by $k^{\mathrm{dcl}}$.

- Every element of $k^{\mathrm{dcl}}$ realizes an isolated type over $k$.
- Every element $a \in k^{\mathrm{dcl}}$ is differentially algebraic over $k$, i.e., there is a nonzero $f \in k\{X\}$, $f(a) = 0$.
  The type $\{f(x) \neq 0 : f \in k\{X\} \setminus k\}$ is non-isolated over $k$.
- $C_{k^{\mathrm{dcl}}} = C_k^{\mathrm{acl}}$, i.e., the only new constants are algebraic over the old constants.
  So, if $C_k$ is algebraically closed $C_k = C_{k^{\mathrm{dcl}}}$.

Skip PV extensions

# Linear Differential Equations

Suppose $f(X)$ is a homogenous linear differential polynomial over $k$ of order $n$.

If $x_1, \ldots, x_n$ are solutions to $f(X) = 0$ that are linearly independent over $C_k$, then $x_1, \ldots, x_n$ are a basis over $C_k$ for all solutions to $f(X) = 0$ in $k$.
We call $x_1, \ldots, x_n$ a *fundamental system of solutions*.
We can find a fundamental system of solutions in $k^{\mathrm{dcl}}$.

For example, suppose $f$ has order two.
First find $x_1$ such that $f(x_1) = 0$ and $x_1 \neq 0$.
Then find $x_2$ such that

$$f(x_1) = 0 \wedge W(x_1, x_2) \neq 0$$

where $W$ is the Wronskian

$$\begin{vmatrix} x_1 & x_1' \\ x_2 & x_2' \end{vmatrix}$$

# Picard-Vessiot extensions

**Definition**

Let $l/k$ be differential fields. We say that $l$ is a *Picard–Vessiot extension* of $k$ if $C_k = C_l$ and there is a homogeneous linear differential equation $f(X) = 0$ and $x_1, \ldots, x_n \in l$ a fundamental system of solutions such that $l = k\langle x_1, \ldots, x_n\rangle$. We say that $l/k$ is a Picard–Vessiot extension for $f$.

# Existence of Picard-Vessiot extensions

## Lemma (Kolchin)

*Let $k$ be a differential field with algebraically closed constant field $C_k$, and let $f(X) = 0$ be a homogeneous linear differential equation over $k$. There is $l/k$ a Picard–Vessiot extension for $f$ with $l$ contained in $k^{\mathrm{dcl}}$. Moreover, if $l_1$ is a second Picard–Vessiot extension of $k$ for $f$, then $l_1$ is isomorphic to $l$ over $k$.*

There is a fundamental system of solutions $x_1, \ldots, x_n \in k^{\mathrm{dcl}}$ and $C_k = C_{k^{\mathrm{dcl}}}$.

Suppose $l_1$ is a second Picard–Vessiot extension of $K$.
Consider $l_1^{\mathrm{dcl}}$. There is an embedding $j : k^{\mathrm{dcl}} \to l_1^{\mathrm{dcl}}$, fixing $k$ and $C_{l_1^{\mathrm{dcl}}} = C_{l_1} = C_k$.

$j(x_1), \ldots, j(x_n)$ is a fundamental system of solutions, then the image of $l$ is $l_1$.

# Uniqueness of differential closure?

## Theorem (Shelah)

*If $K$ is a differential closure of $k$, then $K$ and $k^{\mathrm{dcl}}$ are isomorphic over $k$.*

- **step 1** (Ressayre) If $K, K_1 \models$DCF are constructible over $k$, then $K$ and $K_1$ are isomorphic over $k$.
  (this works for any theory) .
- **step 2** If $K$ is constructible over $k$ and $k \subset L \subset K$, then $L$ is constructible over $k$.
  (this uses stability)
- **step 3** Suppose $K$ is a differential closure of $k$. Then $K$ embeds over $k$ in $k^{\mathrm{dcl}}$, hence $K$ is constructible over $k$ and thus isomorphic to $k^{\mathrm{dcl}}$.

# Minimality of differential closures?

**Theorem (Kolchin/Rosenlicht/Shelah)**

*Differential closures need not be minimal.*
*If $C$ is a field of constants then $C^{\mathrm{dcl}}$ is isomorphic over $C$ to a proper subfield of itself.*

**Definition**

We say that $I \subset k^{\mathrm{dcl}}$ is *indiscernible* over $k$, if for any $x_1, \ldots, x_n, y_1, \ldots, y_n \in I$ with $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ distinct, then

$$f(x_1, \ldots, x_n) = 0 \Leftrightarrow f(y_1, \ldots, y_n) = 0$$

for any $f \in k\{X_1, \ldots, X_n\}$.

# indiscernibles and minimality of differential closure

**Lemma (Sacks)**

*$k^{\rm dcl}$ is minimal over $k$ if and only there is no infinite set $I$ of indiscernible over $k$.*

Suppose $I \subset k^{\rm dcl}$ is an infinite set of indiscernible over $k$. Let $a \in I$ and let $J = I \setminus \{a\}$

Let $K \subseteq k^{\rm dcl}$ be the differential closure of $k\langle J\rangle$.

Clearly $K$ is a differential closure of $k$, thus $K \cong k^{\rm dcl}$.

But $a$ is not isolated over $k\langle J\rangle$.
If $\phi(x)$ is a formula with parameters in $k\langle J\rangle$ satisfied by $a$ then, by indiscernibility, $\phi$ is satisfied by any element of $J$ which doesn't occur as a parameter of $\phi$. Thus $\mathrm{tp}(a/k\langle J\rangle)$ is not isolated.
Thus $a \notin K$.

# Non-minimality of differential closures

Let $C$ be a field of constants. Let $g(X) = \frac{X}{X+1}$. Let
$I = \{x \in C^{\rm dcl} : x' = g(x), x \neq 0\}$.
$I$ is infinite, if $a_1, \ldots, a_n \in I$, then $C^{\rm dcl}$ contains a solution to

$$X' = g(X) = 0 \wedge X \prod (X - a_i) \neq 0$$

is another element of $I$.

We will show $I$ is indiscernible over $C$.
It suffices to show that if $x_1, \ldots, x_n \in I$, then $x_1, \ldots, x_n$ are algebraically independent over $C$.

# Rosenlicht's argument

**Theorem**

*Let $k$ be a differential field with $C_k = C$. Let $g(X) \in C(X)$. Suppose $c_1, \ldots, c_n \in C$, $u_1, \ldots, u_n, v \in C(X)$ such that*

$$\frac{1}{g(X)} = \sum_{i=1}^{n} c_i \frac{du_i/dX}{u_i} + \frac{dv}{dX}.$$

*If $x, y \in k^{\mathrm{dcl}}$, $x' = g(x)$, $y' = g(y)$, $x, y \notin k^{\mathrm{alg}}$ but $x, y$ are algebraically dependent over $k$, then $v(x)' = v(y)'$.*

Apply this when $g(X) = \frac{X}{X+1}$ we use $u = v = X$

$$\frac{X+1}{X} = \frac{dX/dX}{X} + \frac{dX}{dX}$$

# Rosenlicht's argument

$g(X) = \frac{X}{X+1}$, $v(X) = X$.

If $x' = g(x), y' = g(y)$ are nonconstant but algebraically dependent over $C$, then $v(x)' = v(y)'$.

Thus $x' = y'$ and

$$\frac{x}{1+x} = \frac{y}{1+y}$$

and $x = y$.

An inductive argument generalize this to finitely many solutions.

# Other contrasts to algebraically closed fields

• The equation $X' = \frac{X}{1+X}$ can't have a good Galois theory as the automorphism group is the full symmetric group.

• (Counting models) If $\kappa \geq \aleph_1$, there is a unique algebraically closed field of characteristic 0 of cardinality $\kappa$.
There are $2^\kappa$ non-isomorphic differentially closed fields of cardinality $\kappa$.

Idea: Let $A$ be a set of $\kappa$ algebraically independent constants. Let $G$ be a graph on $A$. If $E(a, b)$ add $\kappa$ solutions to $X' = (a + b)\frac{X}{1+X}$, otherwise we will have only $\aleph_0$. (A variant on Rosenlicht's argument says this is possible.) Ideas of Shelah's give $2^\kappa$ models.
A different argument is needed to show there are $2^{\aleph_0}$ nonisomorphic countable DCF.

# Other contrasts to algebraically closed fields–strongly minimal sets

• The Rosenlicht set $I = \{x \neq 0 : x' = \frac{x}{x+1}\}$ is strongly minimal, i.e., any definable subset is finite or co-finite.

In algebraically closed fields $K$, strongly minimal sets are irreducible algebraic curves $X \subset K^n \pm$ finitely many points.
To a model theorist these are all the same as $K$ in that they are interalgebraic with $K$.

In DCF, the constant field $C$ is strongly minimal but very different from $I$.
Understanding strongly minimal sets and their interrelationships in DCF has been one of the main areas of research in the subject over the last 20 years and an important part of many of the most interesting applications.

# Computability?

**Theorem (Harrington)**

*If $k$ is a computable differential field, then there is a computable copy of $k^{\mathrm{dcl}}$.*

$\mathbb{Q}^{\mathrm{dcl}}$ might be a candidate for a "natural" differentially closed field **but** the construction is a nonexplicit priority argument.

The difficulty is that given a formula $\phi(\mathbf{x})$ we need to find an isolated formula that implies it.
No algorithm is know to do this. In Harrington's construct we guess, sometimes find out we are wrong and have to guess again, but eventually guess correctly (even though we never know we've guessed correctly).

Miller, Ovchinnikov and Trushin, have investigate computational questions about $\{(f, g) \in k\{X\} : f(x) = 0 \wedge g(x) \neq 0 \text{ isolates a type over } k\}$ for $k$ a computable field.

Thank you!